Chapter two

In this introductory chapter will study Course description for second course written in renewable energy department.

Contents.

Review of Inverse functions and inverse trigonometric functions.

In every day language the term (inversion) conveys the idea of a reversal. For example in meteorology a temperature inversion is a reversal in the usual temperature proper

Definition:

The function f has inverse (denoted by f^{-1}) if is

onto and one to one function and

$f^{-1}\big(f(x)\big) = f(f^{-1}(x)) = x.$

Example:

Find the inverse of $y = f(x) = 2x + 3.$

Solution:

Since $y = 2x + 3$ is one to one and onto (check)

Then $f^{-1}(x)$ exist.

$x = \frac{y-3}{2} \rightarrow f^{-1}(x) = \frac{x-3}{2}$

$$f(f^{-1}(x)) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = x$$

$\therefore f(f^{-1}(x)) = x.$

$f^{-1}\big(f(x)\big) = x$ (check)

Example:

Find the inverse of $y = x^2, x \geq 0$.

Solution:

Since $y = x^2, x \geq 0$. is one to one and onto (check)

Then $f^{-1}(x)$ exist.

$$x = \sqrt{y} \to f^{-1}(x) = \sqrt{x}$$

$$f(f^{-1}(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(x^2) = \sqrt{x^2} = x.$$

Conform of each of the following.

1-The inverse of $f(x) = 2x\ is\ f^{-1}(x) = \frac{1}{2}x.\ (cheak)$

2-The inverse of $f(x) = x^3\ is\ f^{-1}(x) = x^{\frac{1}{3}}.\ (cheak)$

Inverse Trigonometric functions

A common problem in trigonometry is to find an angle whose trigonometric functions are known.

properties:

$$1 - cscx = \frac{1}{sinx}$$

$$2 - secx = \frac{1}{cosx}$$

$$3 - tanx = \frac{sinx}{cosx}$$

$$4 - cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$$

$$5 - sec^2x - tan^2x = 1$$

$$6 - csc^2x - cot^2x = 1$$

$$7 - tanx\ cotx = 1$$

Theorem:

$$1 - sec^{-1}(x) = cos^{-1}\left(\frac{1}{x}\right)$$

<u>Proof (1):</u>

$$Let\ y = sec^{-1}(x)$$

$$\text{sec}\quad y = \text{sec}\quad sec^{-1}(x)$$

$$\leftrightarrow x = secy$$

$$\leftrightarrow x = \frac{1}{cosy}$$

$$\leftrightarrow cosy = \frac{1}{x}$$

$$cos^{-1}\ cosy = cos^{-1}\frac{1}{x}$$

$$\leftrightarrow y = cos^{-1}\left(\frac{1}{x}\right)$$

$sec^{-1}(x) = cos^{-1}(\frac{1}{x})$

$2 - csc^{-1}(x) = sin^{-1}\left(\frac{1}{x}\right)$ (check)

$3 - sin^{-1}(x) + cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Proof (3):

$$Let\ y = sin^{-1}(x)$$

$$sin\ y = \sin \quad sin^{-1}(x)$$

$$\leftrightarrow x = siny$$

$$\leftrightarrow x = \cos\left(\frac{\pi}{2} - y\right)$$

$$cos^{-1}(x) = cos^{-1} \cos\left(\frac{\pi}{2} - y\right)$$

$$cos^{-1}(x) = \frac{\pi}{2} - y$$

$$\leftrightarrow cos^{-1}(x) + y = \frac{\pi}{2}$$

$$\leftrightarrow cos^{-1}(x) + sin^{-1}(x) = \frac{\pi}{2}$$

$4 - cos^{-1}(-x) = \pi - cos^{-1}(x)$ (check)

$5 - sec^{-1}(-x) = \pi - sec^{-1}(x)$

Proof (5):

$$Let\ y = sec^{-1}(-x)$$

$$sec\ y = \sec \quad sec^{-1}(-x)$$

$$sec\ (y) = -x$$

$$\leftrightarrow x = -sec\ (y)$$

$$\leftrightarrow -x = -sec\ (\pi - y)$$

$$sec^{-1}(-x) = -sec^{-1}sec\ (\pi - y)$$

$$sec^{-1}(-x) = \pi - y$$

$$\leftrightarrow sec^{-1}(-x) = \pi - sec^{-1}(x).$$

$6 - sin\left(sin^{-1}(x)\right) = x \qquad -1 \le x \le 1$

$7 - sin^{-1}\left(sin\ (x)\right) = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$8 - tan^{-1}(x) + cot^{-1}(x) = \frac{\pi}{2}$

Example:

Find the domain and Rang of $sin^{-1}(x)$ and sketch

Solution:

$f(x) = sin^{-1}(x) \leftrightarrow x = siny$

$f(x) = sin^{-1}(x)$ is one to one on $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

$D_f = -1 \le x \le 1.$

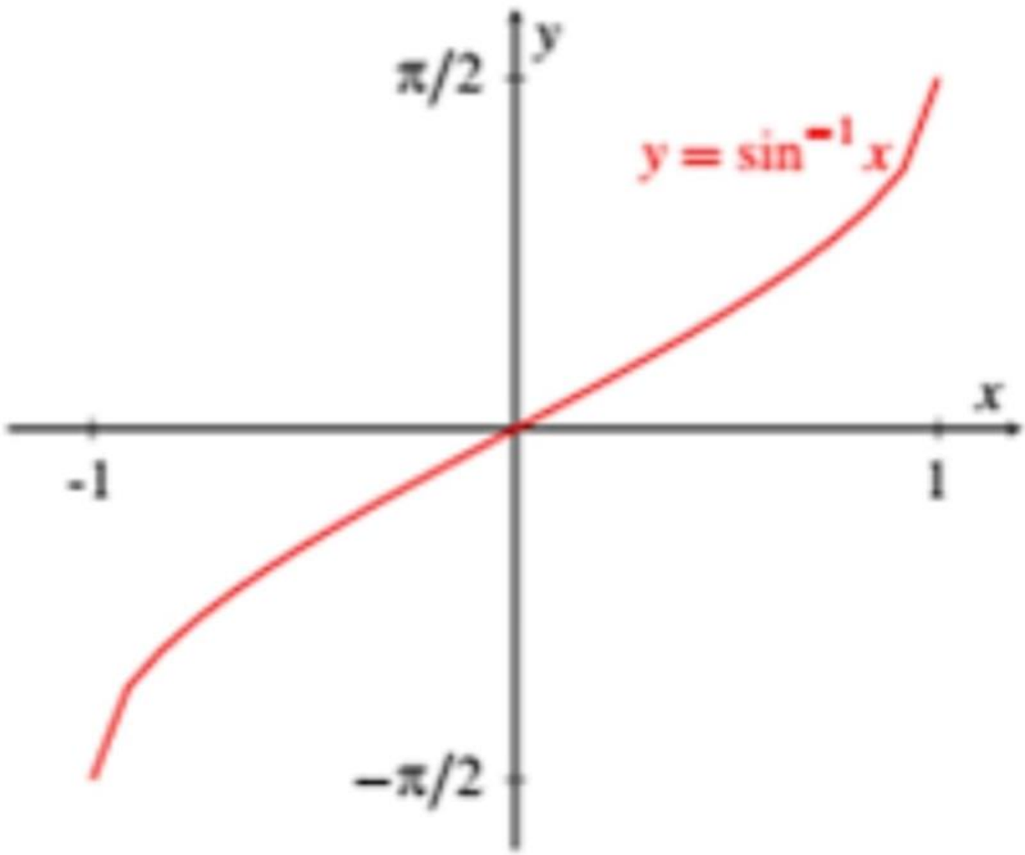
$R_f = \frac{-\pi}{2} \le y \le \frac{\pi}{2}.$

$sin^{-1}(0) = 0$

$sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$sin^{-1}\left(\sqrt{\frac{3}{2}}\right) = \frac{\pi}{3}.$



Properties:

$1 - \quad sin\,(sin^{-1}x) = x$

Remark:

$f(x) = sin^{-1}(-x) = -sin^{-1}(x)$

Proof:

$$y = sin^{-1}(-x)$$

$$siny = \sin \quad sin^{-1}(-x)$$

$$\leftrightarrow -x = siny$$

$$\leftrightarrow x = -siny$$

$$\leftrightarrow x = \sin(-y)$$

$$sin^{-1}(x) = -y$$

$$y = -sin^{-1}(x)$$

$$\therefore \qquad sin^{-1}(-x) = -sin^{-1}(x)$$

Example:

Find the domain and Rang of $cos^{-1}(x)$ and sketch

Solution:

$f(x) = cos^{-1}(x) \leftrightarrow x = cosy$

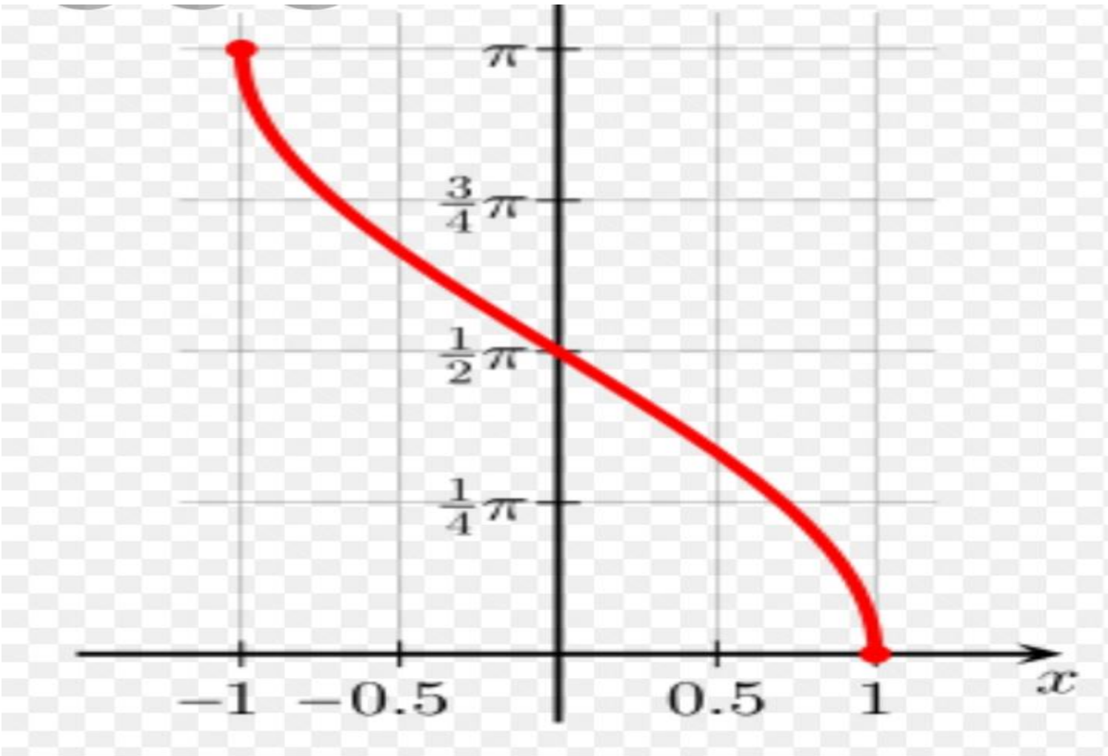
$f(x) = cos^{-1}(x)$ is one to one on $[0, \pi]$

$D_f = -1 \leq x \leq 1.$

$R_f = 0 \le y \le \pi.$

$* \; sin^{-1}\dfrac{1}{x} = cos^{-1}x$

$* \; cos\,(cos^{-1}x) = x$



$* \; cos^{-1}\dfrac{1}{x} = sec^{-1}x$

Example:

Find the domain and Rang of $tan^{-1}(x)$ and sketch

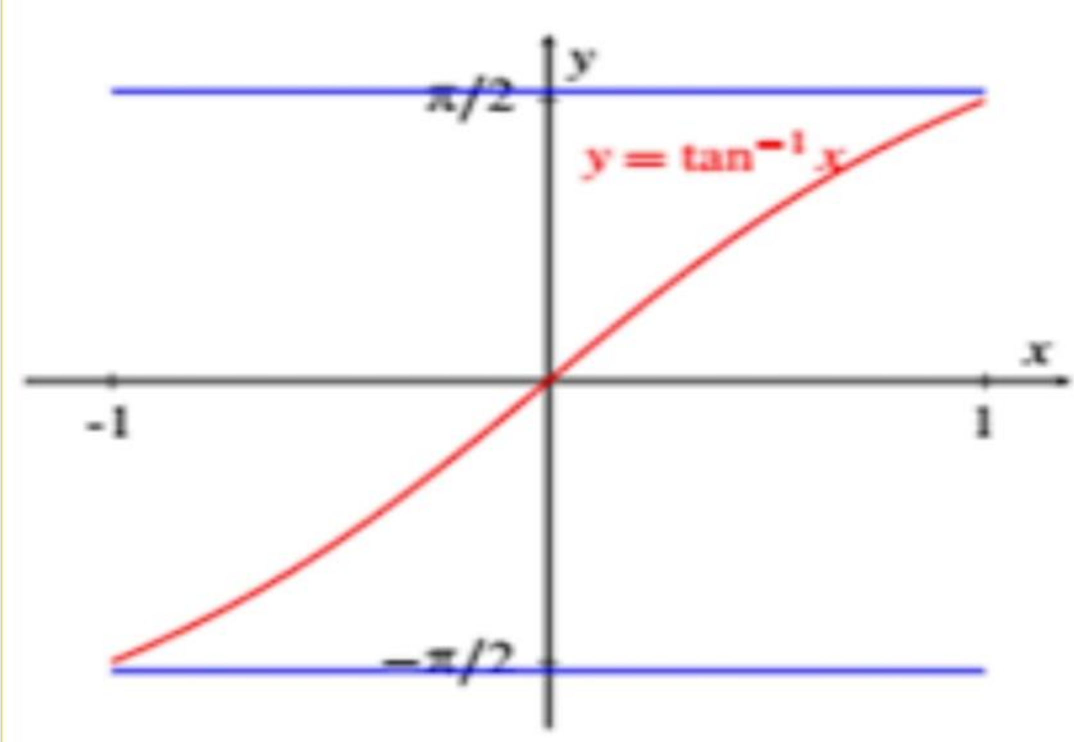
Solution:

$f(x) = tan^{-1}(x) \leftrightarrow x = tany$

$f(x) = tan^{-1}(x)$ is one to one on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$D_f = -\infty \leq x \leq \infty.$

$R_f = \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}.$



Remark:

$\boldsymbol{f(x) = tan^{-1}(-x) \leftrightarrow -tan^{-1}(x)}$

Proof:

$$y = tan^{-1}(-x)$$

$$\color{red}{\tan}\quad y = \color{red}{\tan}\quad tan^{-1}(-x)$$

$$\leftrightarrow -x = tany$$

$$\leftrightarrow x = -\tan y$$

$$\leftrightarrow x = \tan(-y)$$

$\leftrightarrow tan^{-1}(x) = -y$

$\therefore -tan^{-1}(x) = -tan^{-1}(-x)$

Remark:

$$tan^{-1}\frac{1}{x} = cot^{-1}x$$

Example:

Find the domain and Rang of $cot^{-1}(x)$ and sketch

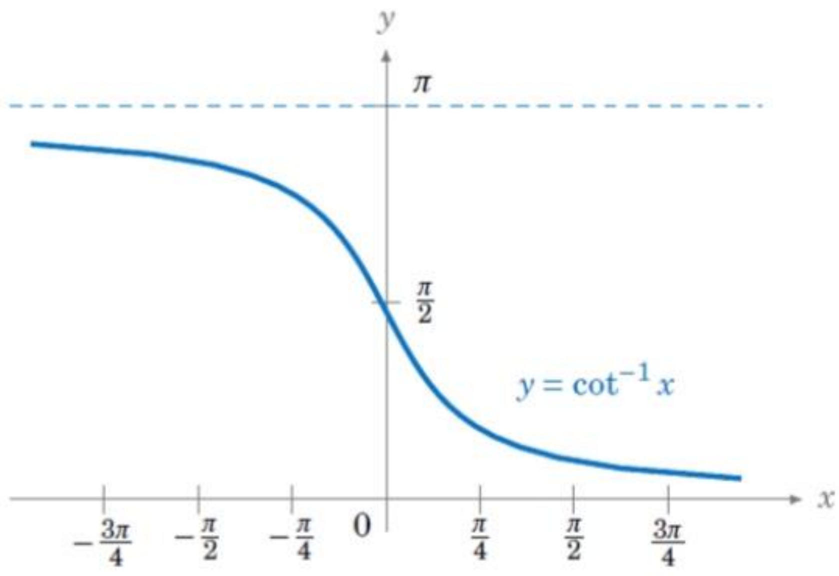
Solution:

$f(x) = tan^{-1}(x) \leftrightarrow x = coty$

$f(x) = cot^{-1}(x)$ is one to one on $(0, \pi)$

$D_f = R \; or \; -\infty \leq x \leq \infty.$

$R_f = (0, \pi).$



Example:

Find the domain and Rang of $sec^{-1}(x)$ and sketch

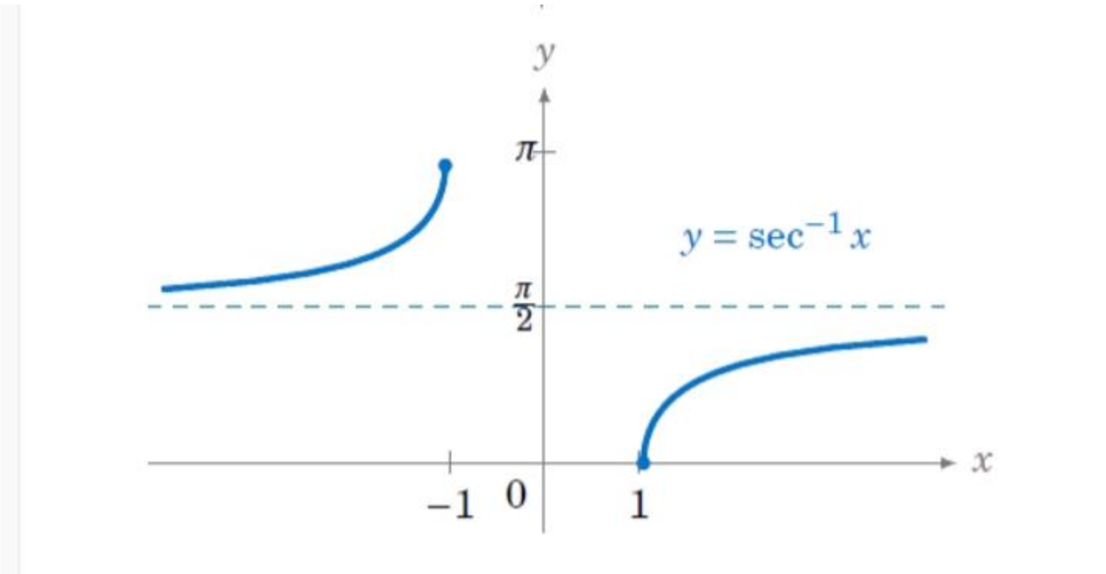
Solution:

$f(x) = sec^{-1}(x) \leftrightarrow x = secy$

$f(x) = sec^{-1}(x)$ is one to one on $[0, \pi]$

$D_f = \{x: x \geq 1\} \cup \{x: x \leq -1\}$

$R_f = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \frac{\pi}{2}.$



Example:

Find the domain and Rang of $csc^{-1}(x)$and and sketch

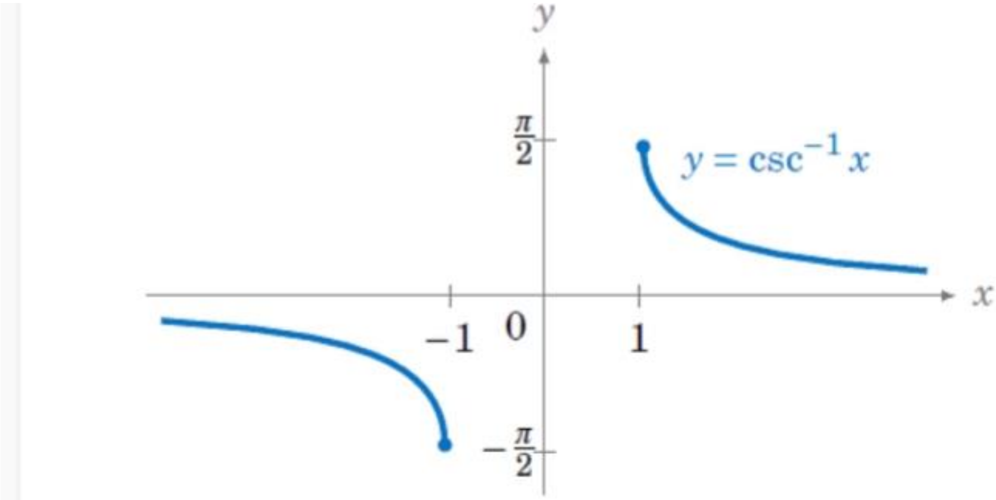
Solution:

$f(x) = csc^{-1}(x) \leftrightarrow x = cscy$

$f(x) = csc^{-1}(x)$ is one to one on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \{0\}$.

$D_f = \{x : x \geq 1\} \cup \{x : x \leq -1\}$

$R_f = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \{0\}$.



Derivative of inverse Trigonometric functions.

Here we will use implicit differentiation to obtain the derivative formula for $y = sin^{-1}u, cos^{-1}u, tan^{-1}u,$ $cot^{-1}u, sec^{-1}u$ and $csc^{-1}u$.

$$1 - \frac{d}{du} sin^{-1}u = \frac{du}{\sqrt{1-u^2}} \qquad (-1 < u < 1).$$

$$2 - \frac{d}{du} cos^{-1}u = \frac{-du}{\sqrt{1-u^2}} \qquad (-1 < u < 1).$$

$$3-\frac{d}{du}tan^{-1}u=\frac{du}{1+u^2}$$

$$4-\frac{d}{du}cot^{-1}u=\frac{-du}{1+u^2}$$

$$5-\frac{d}{du}sec^{-1}u=\frac{du}{|u|\sqrt{u^2-1}} \qquad |u|>1.$$

$$6-\frac{d}{du}csc^{-1}u=\frac{-du}{|u|\sqrt{u^2-1}} \qquad |u|>1.$$

Example:

$find\ \frac{dy}{dx}$ for the function $y=sin^{-1}(x)$

Solution:

$$By \quad \frac{d}{du}sin^{-1}u=\frac{du}{\sqrt{1-u^2}} \qquad (-1<u<1).$$

$$\frac{dy}{dx}sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$$

Example:

$find\ \frac{dy}{dx}$ for the function $y=sin^{-1}\frac{1}{x^3}$.

Solution:

By $\frac{d}{du}sin^{-1}u=\frac{du}{\sqrt{1-u^2}} \qquad -1<u<1.$

$$\frac{dy}{dx}sin^{-1}x^{-3}=\frac{-3x^{-4}}{\sqrt{1-(x^{-3})^2}}$$

Example:

$if\ y = cos^{-1}secx \quad find\ \frac{dy}{dx}$

Solution:

$$By: \frac{d}{du}cos^{-1}u = \frac{-du}{\sqrt{1-u^2}}$$

$$\frac{d}{du}cos^{-1}secx = \frac{-secx\ tanx}{\sqrt{1-(secx)^2}}$$

Example:

$find\ \frac{dy}{dx}$ if $y = cos^{-1}\left(\frac{1}{x}\right)$

Solution:

$$\frac{dy}{dx}cos^{-1}\left(\frac{1}{x}\right) = \frac{-}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(\frac{1}{x^2}\right)$$

Example:

$find\ \frac{dy}{dx}$ if $y = tan^{-1}cscx$

Solution:

By: $\frac{d}{du}tan^{-1}u = \frac{du}{1+u^2}$

$$\frac{dy}{dx}tan^{-1}cscx = \frac{-cscx\ cotx}{1+(cscx)^2}$$

Example:

$find\ \frac{dy}{dx}$ if $y = cot^{-1}(x+1)$

Solution:

By: $\frac{d}{du}cot^{-1}u = \frac{-du}{1+u^2}$

$$\frac{dy}{dx}cot^{-1}(x+1) = \frac{-1}{1+(x+1)^2}$$

Example:

$find\ \frac{dy}{dx}$ if $y = cot^{-1}\,x^2 \quad cosx$

Solution:

By: $\frac{d}{du}cot^{-1}u = \frac{-du}{1+u^2}$

$$\frac{dy}{dx}cot^{-1}\,x^2 \quad cosx = cot^{-1}\,x^2(-sinx) + cosx\,\frac{-2x}{1+(x^2)^2}$$

Example:

$find\ \frac{dy}{dx}$ if $y = sec^{-1}(e^{tanx})$

Solution:

By: $\frac{d}{du}sec^{-1}u = \frac{du}{|u|\sqrt{u^2-1}} \qquad |u| > 1.$

$$\frac{dy}{dx}sec^{-1}(e^{tanx}) = \frac{(secx)^2\ e^{tanx}}{|e^{tanx}|\sqrt{(e^{tanx})^2-1}}$$

Example:

$find\ \frac{dy}{dx}$ if $y = csc^{-1}(cscx)$

Solution:

By: $\frac{d}{du} csc^{-1}u = \frac{-du}{|u|\sqrt{u^2-1}}$ $|u| > 1.$

$$\frac{dy}{dx} csc^{-1}(cscx) = \frac{cscx\ cotx}{|cscx|\sqrt{(cscx)^2-1}}$$

Example:

$find\ \frac{dy}{dx}$ if $y = sin(tan^{-1}x^3)$

Solution:

$$\frac{dy}{dx} sin(tan^{-1}x^3) = cos(tan^{-1}x^3).\frac{\mathbf{3x^2}}{\mathbf{1}+(x^3)^2}$$

Exercise:

$1 - y = sin^{-1}\left(x^3 - \frac{1}{x}\right)$ $find\ \frac{dy}{dx}$

2-$find\ \frac{dy}{dx}$ if $y = [cos^{-1}(2x^2+3)]^3$

3-$find\ \frac{dy}{dx}$ if $y = tan^{-1}(sinx + \cos x)$

4-$find\ \frac{dy}{dx}$ if $y = sec^{-1}(x\sqrt{cosx})$

$5 - find\ \frac{dy}{dx}$ if $y = sec(tan^{-1}\frac{1}{x})$

$6 - find\ \frac{dy}{dx}$ if $y = cot(tan^{-1}\sqrt{x})$

$7 - find\ \frac{dy}{dx}$ if $y = tan(tan^{-1}2x)$

2-Hyperbolic functions

In this section we will study certain combination of $e^x \ and \ e^{-x}$ called hyperbolic functions these functions which arise in various engineering applications have many properties in common with the trigonometric functions.

Definition:

Hyperbolic sine is → $sinhx = \frac{e^x - e^{-x}}{2}$

Hyperbolic cosine is → $coshx = \frac{e^x + e^{-x}}{2}$

Hyperbolic tangent is → $tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent is → $cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant is → $sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$

Hyperbolic cosecant is → $cschx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}$

Example:

Find sinh x if x=0.

Solution.

$$sinh0 = \frac{e^x - e^{-x}}{2} = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.$$

Example:

Find cosh x if x=0.

Solution:

$$cosh0 = \frac{e^x + e^{-x}}{2} = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1.$$

Example:

Find sinh x if x=2.

Solution:

$$sinh2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269.$$

Example:

Find the Hyperbolic secant where the angle is π

Solution:

Hyperbolic secant is → $sech\pi = \frac{1}{cosh\pi} = \frac{2}{e^{\pi}+e^{-\pi}}$

3-Derivative Hyperbolic functions

The derivative formulas for Hyperbolic functions can be obtain by expressing.

Theorem.

$$1-\frac{d}{dx}sinhx = coshu\frac{du}{dx}$$

$$2-\frac{d}{dx}coshu = sinhu\frac{du}{dx}$$

$$3-\frac{d}{dx}tanhu = sech^2u\frac{du}{dx}$$

$$4-\frac{d}{dx}cothu = -csch^2u\frac{du}{dx}$$

$$5-\frac{d}{dx}sechu = -sechu\ tanh\,u\frac{du}{dx}$$

$$6-\frac{d}{dx}cschu = -\operatorname{csch}u\coth u\frac{du}{dx}$$

Remark:

$$cosh^2x\ - sinh^2x = 1$$

Example:

Derivative formula tanhx can be obtained by formulas for Hyperbolic functions

Solution.

$$\frac{d}{dx}tanhx = \frac{d}{dx}\frac{sinhx}{coshx}$$

$$= \frac{coshx\ \frac{d}{dx}sinhx - sinhx\frac{d}{dx}coshx}{cosh^2x}$$

$$\frac{cosh^2x\ - sinh^2x}{cosh^2x} = \frac{1}{cosh^2x} = sech^2x.$$

<u>Example:</u>

Derivative formula coshx^3

Solution.

$$\frac{d}{dx}\cosh x^3 = 3x^2\ sinhx^3$$

<u>Example:</u>

Derivative formula sinhx can be obtained by expressing these functions in terms of e^x and e^{-x}.

Solution.

$$\frac{d}{dx}sinhx = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = coshx$$

<u>Example:</u>

Derivative formula coshx can be obtained by expressing these functions in terms of e^x and e^{-x}

Solution:

$$\frac{d}{dx}coshx = \frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{e^{x}-e^{-x}}{2} = sinhx$$

Exercise:

Derivative formula for $cothx$ can be obtained by expressing these functions in terms of e^{x} and e^{-x}

Exercise:

Find the derivative formula for the functions.

$1 - \mathrm{y} = \ln \tanh\mathrm{x}$

$2 - y = seche^{3x}$

$4 - \mathrm{y} = \dfrac{\tanh\mathrm{x}}{sech30}$

3-Inverse Hyperbolic functions and their derivative

The following theorems list the generalized derivative formula for the Inverse Hyperbolic functions.

Theorems.

$$1 - \frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt[2]{1+u^{2}}}\frac{du}{dx}$$

$$2 - \frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt[2]{u^{2}-1}}\frac{du}{dx}. \quad u > 1.$$

$$3-\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}. \qquad |u|<1.$$

$$4-\frac{d}{dx}\coth^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}. \qquad |u|>1.$$

$$5-\frac{d}{dx}\operatorname{sech}^{-1}u=-\frac{1}{u\sqrt[2]{1-u^2}}\frac{du}{dx}. \qquad 0<u<1.$$

$$6-\frac{d}{dx}\operatorname{csch}^{-1}u=-\frac{1}{|u|\sqrt[2]{1+u^2}}\frac{du}{dx}. \qquad u\neq 0.$$

Example:

Find the derivative formula for $tanh^{-1}x^2$

Solution:

$$By: \frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}. \qquad |u|<1.$$

$$\frac{d}{dx}\tanh^{-1}x^2=\frac{2x}{1-(x^2)^2}.$$

Example:

Find the derivative formula for $\operatorname{csch}^{-1}\frac{1}{x}$

Solution:

$$By: \frac{d}{dx}\operatorname{csch}^{-1}u=-\frac{1}{|u|\sqrt[2]{1+u^2}}\frac{du}{dx}. \qquad u\neq 0.$$

$$\frac{d}{dx}\operatorname{csc}h^{-1}\frac{1}{x} = -\frac{\frac{-1}{x^2}}{\left|\frac{1}{x}\right| \sqrt[2]{1+\left(\frac{1}{x}\right)^2}}$$

Example:

Find the derivative formula for $\operatorname{cot}h^{-1}cosx$

Solution:

$$By\colon \frac{d}{dx}\operatorname{cot}h^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}. \qquad |u| > 1.$$

$$\frac{d}{dx}\operatorname{cot}h^{-1}cosx = \frac{-sinx}{1-(cosx)^2}$$

Exercise:

Find the derivative formula for the functions.

$1 - \mathrm{y} = \operatorname{tan}h^{-1}\left(\frac{2}{3+x}\right)$

$2 - \mathrm{y} = \operatorname{csc}h^{-1}secx^3$

$3 - \mathrm{y} = \operatorname{cos}h^{-1}tanx$

4-Integration involving inverse trigonometric functions.

Let F(x) and f(x) be two functions related as

$$\frac{\mathrm{d}}{dx}F(x) = f(x)$$

Then f(x) is called the derivative of F(x)

f(x) is called an infinite integral of F(x) and denoted by

$$f(x) = \int F(x)dx.$$

$$\int u^n \frac{du}{dx} = \frac{u^{n+1}}{n+1} + c$$

Remark:

$\frac{d}{dx}x^2 = 2x. \qquad \rightarrow \int 2xdx = x^2 + c$

But

$$\frac{d}{dx}(x^2 + 3) = 2x. \quad \rightarrow \int 2xdx = x^2 + c$$

C is called a constant of integration.

Example:

Find $\int \frac{1}{5}x^3dx$

Solution.

$$\int \frac{1}{5}x^3dx = \frac{1}{5}\int x^3dx = \frac{1}{5}\frac{x^4}{4} + c = \frac{x^4}{20} + c$$

Example:

Find $\int \frac{1}{5} x^{-3} dx$

Solution.

$$\int \frac{1}{5} x^{-3} dx = \frac{1}{5} \int x^{-3} dx = \frac{x^{-2}}{-10} + c$$

Example:

Find $\int (\frac{1}{4x^4} + 2\,) dx$

Solution.

$$\int (\frac{1}{4x^4} + 2\,) dx = \int \frac{1}{4x^4} dx + \int 2 \ dx$$

$$= \int \frac{x^{-4}}{4} dx + \int 2 \ dx$$

$$= \frac{1}{4} \int x^{-4} dx + \int 2 \ dx$$

$$= \frac{x^{-3}}{-12} + 2x + c$$

Example:

Find $\int (x^2 + 2\,)^2 dx$

Solution.

$$\int (x^2 + 2)^2 dx = \int \quad (x^4 + 4x^2 + 4)dx$$

$$= \frac{x^5}{5} + \frac{4x^3}{3} + 4x + c$$

Example:

Find $\int (x^2 + 2)^2 \quad 2x \ dx$

Solution.

$$\int (x^2 + 2)^2 \quad 2x \ dx = \frac{(x^2 + 2)^3}{3} + c$$

5- Trigonometric Integrals

We will discuss methods for integrating other kinds of integrals that involve Trigonometric Integrals.

Theorems:

$$1 - \int sinu \frac{du}{dx} dx = -cosu + c$$

$$2 - \int cosu \ \frac{du}{dx} dx = sinu + c$$

$$3 - \int sec^2u \frac{du}{dx} dx = tanu + c$$

$$4 - \int csc^2u \frac{du}{dx} dx = -cotu + c$$

$$5-\int secu\ tanu\frac{du}{dx}dx = secu + c$$

$$6-\int cscu\ cotu\frac{du}{dx}dx = -cscu + c$$

Example:

Evaluate $\int sin\frac{1}{2}xdx$

Solution:

$$\int sin\frac{1}{2}xdx$$

$$2\int\frac{1}{2}sin\frac{1}{2}xdx = -2cos\frac{1}{2}x + c$$

Example:

Evaluate $\int x^2e^{x^3}cose^{x^3}dx$

Solution.

$$\frac{1}{3}\int 3x^2e^{x^3}\ cose^{x^3}dx = \frac{1}{3}sine^{x^3} + c$$

Exercise:

Find the integral formula for the functions.

$$1-\int \frac{1}{x^3}cscx^{-2}\ cotx^{-2}dx$$

$$2-\int cosx\ csc^2\ sinx\ dx$$

Integration involving Exponential

$$\int e^u \, du = e^u + \boldsymbol{c}.$$

Example:

Find $\int e^{2x} \; 2 \; dx$

Solution.

$$\int e^{2x} \; 2 \; dx = e^{2x} + c$$

Example:

Find $\int e^{cosx} \;\; sinx \, dx$

Solution.

$$-\int e^{cosx} \; (-sinx)dx = -e^{cosx} + c$$

Exercise:

Find the integral formula for the functions.

$1 - \int e^{e^x} \;\; e^x dx$

$1 - \int e^{(x^2+4x)^2} \;\; (x^2 + 4x)^2(2x + 4\,)dx$

Integration involving logarithm

$$\int \frac{1}{u} du = ln|u| + c.$$

Example:

Find $\int \frac{x^2}{x^3 + 10} \quad dx$

Solution.

$$\frac{1}{3}\int \frac{3x^2}{x^3 + 10} \; dx = \frac{1}{3} Ln|x^3 + 10| + c$$

Example:

Find $\int tanx \quad dx$

Solution.

$$\int tanx \; dx = -\int \frac{-sinx}{cosx} \quad dx = -Ln|cosx| + c$$

Example:

Find $\int e^{ln\, secx} \quad dx$

Solution.

Find $\int e^{ln\, secxtanx} \quad dx = \int secxtanx\, dx = secx + c$

Example:

Find $\int Lne^{x^{-3}} \quad dx$

Solution.

$$\int Lne^{x^{-3}} \quad dx = \int x^{-3}dx = \frac{x^{-2}}{-2} + c$$

Example:

$$find \int \frac{3x^2}{(x^3+10)^5} \; dx$$

Solution.

$$\int \frac{3x^2}{(x^3+10)^5} \; dx = \int 3x^2 \, (x^3+10)^{-5} \, dx = \frac{(x^3+10)^{-4}}{-4}$$

Exercise:

Find the integral formula for the functions.

$1 - \int \frac{cosx}{e^{cosx}} \; dx$

$2 - \int \frac{x^3}{x^4} dx$

$3 - \int 0 \, dx$

Integration inverse trigonometric functions.

We will derive some related integration formulas that involve Inverse Trigonometric functions.

Theorem.

$$1-\int \frac{du}{\sqrt{1-u^2}} = sin^{-1} u + c$$

$$2-\int \frac{-du}{\sqrt{1-u^2}} = cos^{-1} u + c$$

$$3-\int \frac{du}{1+u^2} = tan^{-1} u + c$$

$$4-\int \frac{-du}{1+u^2} = cot^{-1} u + c$$

$$5-\int \frac{du}{u\sqrt{u^2-1}} = sec^{-1} |u| + c$$

$$6-\int \frac{-du}{u\sqrt{u^2-1}} = csc^{-1} u + c$$

Example:

Evaluate $\int \frac{dx}{1+25x^2}$

Solution.

$$By: \int \frac{du}{1+u^2} = tan^{-1} u + c$$

substituting

$u = 5x, \qquad du = 5\,dx.$

$yields$

$$\int \frac{dx}{1+25x^2} = \frac{1}{5}\int \frac{5dx}{1+(5x)^2} = \frac{1}{5} tan^{-1}(5x) + c$$

Example:

Evaluate $\int \frac{dy}{y\sqrt{49y^2-1}}$

Solution.

By: $\int \frac{du}{u\sqrt{u^2-1}} = sec^{-1}|u| + c$

substituting

$u = 7y, \qquad du = 7\,dy.$

$yields$

$$\int \frac{dy}{y\sqrt{49y^2-1}} = \frac{1}{7}\int \frac{7dy}{y\sqrt{(7y)^2-1}} = \frac{1}{7} sec^{-1}(7y) + c$$

Example:

Evaluate $\int \frac{-\,dx}{1+3x^2} =$

Solution.

By: $\int \frac{du}{1+u^2} = cot^{-1} u + c$

substituting

$u = \sqrt{\mathbf{3}}x, \qquad du = \sqrt{\mathbf{3}}\,dx.$

$yields$

$$\int \frac{-dx}{1+3x^2} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\int \frac{-\sqrt{3}dx}{1+\left(\sqrt{\mathbf{3}}x\right)^2} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}cot^{-1}\sqrt{\mathbf{3}}x + c$$

Example:

Evaluate $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

Solution.

$$By \int \frac{du}{\sqrt{1-u^2}} = sin^{-1} u + c$$

substituting

$u = \boldsymbol{e^x}, \qquad du = \boldsymbol{e^x}\, dx.$

$yields$

$$\int \frac{e^x dx}{\sqrt{1-e^{2x}}} = \int \frac{e^x dx}{\sqrt{1-(e^x)^2}} = sin^{-1}(e^x) + c.$$

Exercise:

Find the integral formula for the functions.

$$1 - \int \frac{-cosx}{\sqrt{1-sin^2x}}dx$$

$$2 - \int \frac{-1}{\sqrt{1-\left(\frac{1}{5}x\right)^2}}dx$$

$$3-\int \frac{\frac{1}{x}}{1+lnx^2}dx$$

$$4-\int \frac{-z^{\frac{-2}{3}}}{1+z^{\frac{2}{3}}}dz$$

$$5-\int \frac{x}{x^2\sqrt{x^4-1}}dx$$

$$6-\int \frac{-1}{3y\sqrt{9y^2-1}}dy$$

Integration by parts

In this section we will discuss an integration technique that is essentially an anti derivative formulation of the formula for differentiating a product of two functions.

Definition:

The integration dv is the product of the algebraic u function.

$$\int udv = u\,v - \int vdu$$

Where.

$u \to du$

$$\int dv = v$$

Example:

Evaluate $\int x\, e^x\, dx$.

Solution.

$u = \mathrm{x} \quad \rightarrow du = 1.dx$

$$dv = e^x dx \quad \rightarrow \int dv = \int e^x dx \rightarrow v = e^x$$

$By\!: \int u dv = u\, v - \int v du$

$$\int xe^x\, dx = xe^x - \int e^x\, dx$$

$$\int xe^x\, dx = xe^x - e^x + c.$$

Example:

Evaluate $\int x\, sinx\, dx$.

Solution.

$u = \mathrm{x} \quad \rightarrow du = 1.dx$

$$dv = sinx\, dx \quad \rightarrow \int dv = \int sinxdx \rightarrow v = -cosx$$

$By\!: \int u\, dv = u\, v - \int v du$

$$\int xsinx\, dx = xcosx - \int -cosx\, dx$$

$$\int xsinx\,dx = xcosx + sinx + c$$

Example:

Evaluate $\int x^2 sinx\,dx$.

Solution.

$u = x^2 \quad \rightarrow du = 2x\,dx$

$dv = sinxdx \quad \rightarrow \int dv = \int sinxdx \rightarrow v = -cosx$

By: $\int udv = u\,v - \int vdu$

$$\int x^2 sinx\,dx = -x^2 cosx + \int 2x \quad cosx\,dx \qquad *$$

$$\int 2x\,cosx\,dx = ?$$

$u = 2\text{x} \quad \rightarrow du = 2\,dx$

$dv = cosxdx \quad \rightarrow \int dv = \int cosxdx \rightarrow v = sinx$

Now:

$$\int 2x\,cosx\,dx = u\,v - \int vdu$$

$$\int 2x\,cosx\,dx = -2xsinx - 2\int sinxdx$$

$$\int 2x\,cosx\,dx = -2xsinx + 2cosx + c$$

By *

$$\int x^2 sinx\, dx = x^2 cosx + \int 2x\, cosx\, dx \qquad *$$

$$\int x^2 sinx\, dx = x^2 cosx + (-2xsinx + 2cosx + c)$$

$$\int x^2 sinx\, dx = x^2 cosx - 2xsinx + 2cosx + c$$

Example:

Evaluate $\int lnx\, dx$.

Solution.

$$u = \ln x \quad \rightarrow du = \frac{1}{x} dx$$

$$dv = dx \quad \rightarrow \int dv = \int dx \rightarrow v = x$$

By: $\int udv = u\, v - \int vdu$

$$\int lnx\, dx = xlnx - \int x \frac{1}{x} dx$$

$$\int lnx\, dx = xlnx - \int dx$$

$$\int lnx\, dx = xlnx - x + c$$

Example:

Evaluate $\int x\, csc^2 x dx$.

Solution.

$$u = \text{x} \quad \rightarrow du = 1. dx$$

$$dv = csc^2 x dx \;\; \rightarrow \int dv = \int csc^2 x dx \rightarrow v = -cotx$$

By: $\int udv = u\,v - \int vdu$

$$\int xcsc^2 x\; dx = -xcotx + \int \; cotx\; dx$$

$$\int xcsc^2 x\; dx = -xcotx + ln|sinx| + c$$

Exercise:

$1 - \int x^2\, e^x dx$

$2 - \int x\, cosxdx$

$3 - \int cosx\, e^x dx$

6-Trigonometric substitution

In this section we will discuss a method for evaluating integrals containing radicals by making substitutions involving Trigonometric functions.

Case one:

We will connected with integrals that contain expressions of the form

$\sqrt{a^2 - x^2}$

In which a is a positive constant

$x = a\ sin\theta, \qquad {}^{-\pi}/_{2} \le \theta \le {}^{\pi}/_{2}$

Which yields

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 sin^2\theta} = \sqrt{a^2(1 - sin^2\theta)}$$

$$sin^2\theta + cos^2\theta = 1$$

$$a\sqrt{cos^2\theta} = a|cos\theta| = acos\theta,$$

$$cos\theta \ge 0 \qquad since \qquad {}^{-\pi}/_{2} \le \theta \le {}^{\pi}/_{2}$$

Example (*):

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4 - x^2}}$

Solution.

To eliminate the radical we make the substitution

$$x = 2\ sin\theta \rightarrow sin\theta = \frac{x}{2}$$

$$x = 2\ sin\theta, \qquad dx = 2cos\theta\ d\theta.$$

This yields

$$\int \frac{dx}{x^2\sqrt{4 - x^2}} = \int \frac{2cos\theta\ d\theta}{(2\ sin\theta)^2\sqrt{4 - (2\ sin\theta)^2}}$$

$$\int \frac{2cos\theta\ d\theta}{(2\ sin\theta)^2(2cos\theta)} = \int \frac{d\theta}{(2\ sin\theta)^2} = \frac{1}{4}\int \frac{d\theta}{sin^2\theta}$$

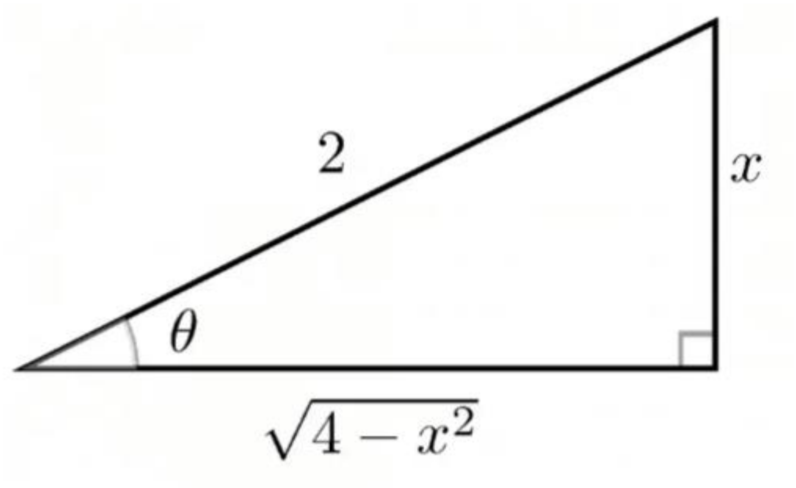
$$\frac{1}{4}\int csc^2\theta d\theta = -\frac{1}{4}cot\theta + c \qquad (1)$$

At this point we have completed the integration.

$$cot\theta = \frac{\sqrt{4 - x^2}}{x}$$

Substituting this in (1) yields

$$\int \frac{dx}{x^2\sqrt{4 - x^2}} = -\frac{1}{4}\frac{\sqrt{4 - x^2}}{x} + c$$



Example:

Evaluate $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4 - x^2}}$

Solution.

We can use the result in example (*) with the x-limit of integration yields.

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = \frac{\sqrt{3}-1}{4}$$

Case two:

We will connected with integrals that contain expressions of the form

$$\sqrt{a^2+x^2}$$

In which a is a positive constant

$x = a\, tan\theta, \qquad {}^{-\pi}/_2 \le \theta \le {}^{\pi}/_2$

$x = a\, tan\theta, \qquad dx = a\, sec^2\theta\, d\theta.$

Example:

Evaluate $L = \int_0^{\frac{\pi}{4}} \sqrt{1+x^2}\, dx$

Solution.

We can use the case two.

$x = tan\theta, \qquad dx = sec^2\theta\, d\theta.$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1+tan^2\theta}\quad sec^2\theta\, d\theta$$

$$\int_0^{\frac{\pi}{4}} \sqrt{sec^2\theta}\;\; sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} sec\theta\;\; sec^2\theta\, d\theta$$

$$\int_0^{\frac{\pi}{4}} sec^3\theta\, d\theta = \left[\frac{1}{2} sec\theta tan\theta + \frac{1}{2} ln|sec\theta + tan\theta|\right]_0^{\frac{\pi}{4}}$$

$$\frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right] \approx 1.148$$

Remark:

$$1 - \int sec^n\, xdx = \frac{sec^{n-2} x tanx}{n-1} + \frac{n-2}{n-1}\int sec^{n-2} xdx$$

$$2 - \int sec\quad xdx = ln|secx + tanx| + c$$

Case three:

We will connected with integrals that contain expressions of the form

$$\sqrt{x^2 - a^2}$$

In which a is a positive constant

$x = \ asec\theta,\qquad 0 \le \theta \le {}^{\pi}/_{2}$

$$x = a\ sec\theta, \qquad\qquad dx = a\ sec\theta tan\theta\ d\theta.$$

Example:

Evaluate $\int \frac{\sqrt{x^2-25}}{x} dx$

Solution.

We can use the case three.

$x = 5\,secx, \qquad dx = 5\,secxtanx\,dx.$

$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{25sec^2x-25}}{5secx}5\,secxtanx\,dx$$

$$\int \frac{5|tanx|}{5secx}5\,secxtanx\,dx$$

$$5\int tan^2xdx$$

$$5\int (sec^2x-1)dx = 5tanx - 5x + c$$

6-Integrating rational functions by partial fractions

In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions tat can be integrated by the methods studied in earlier sections.

Linear factor rule.

$$1 - \frac{1}{(x)^2} = \frac{A}{x} + \frac{B}{x^2}$$

$$2 - \frac{1}{(x)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{c}{x^3}$$

$$3 - \frac{1}{2x^3-8x} = \frac{1}{2x\,(x^2-4)}$$

$$= \frac{1}{2x\,(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{c}{x+2}$$

Example:

Evaluate

$$\int \frac{x+1}{x^2-3x+2}\,dx$$

Solution.

$$\frac{x+1}{x^2-3x+2}=\frac{x+1}{(x-1)(x-2)}=\frac{A}{X-1}+\frac{B}{x-2}$$

Multiply both side by the denominator $(x-1)(x-2)$

$$x+1=A(x-2)+B(x-1) \qquad ---(1)$$

Let $x-2=0 \rightarrow [x=2]$.

By substitute in (1)

$$3=A(0)+B(1)\rightarrow [B=3]$$

Let $x-1=0 \rightarrow [x=1]$.

By substitute in (1)

$$2=A(-1)+B(0)\rightarrow [A=-2]$$

Now

$$\int \frac{x+1}{x^2-3x+2}\,dx=\int \frac{A}{X-1}\,dx+\int \frac{B}{x-2}\,dx$$

$$\int \frac{x+1}{x^2-3x+2}\,dx=-2\int \frac{1}{X-1}\,dx+3\int \frac{1}{x-2}\,dx$$

$$\int \frac{x+1}{x^2-3x+2}dx = -2\ln|x-1| + 3\ln|x-2| + c.$$

Example:

Evaluate $\int \frac{x^2}{(x+1)(x-1)^2}dx$

Solution:

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{c}{(x-1)^2}$$

Multiply both side by the denominator

$(x+1)(x-1)(x-1)^2$

$$x^2 = A(x-1)^2 + B(x+1)(x-1) + c(x+1) --(1)$$

Let $x-1=0 \rightarrow x=1.$

By substitute in (1)

$$1 = A(0) + B(0) + c(2) \rightarrow \left[c = \frac{1}{2}\right] \quad ---1$$

Let $x+1=0 \rightarrow x=-1.$

By substitute in (1)

$$1 = A(4) + B(0) + c(0) \rightarrow \left[A = \frac{1}{4}\right] ----2$$

Now: By equation (1) and (2) we rusult

1=A+B

$$B = 1 - A = 1 - \frac{1}{4} \rightarrow B = \frac{3}{4}$$

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{c}{(x-1)^2}$$

$$\frac{x^2}{(x-1)(x-1)^2} = \frac{1}{4}\frac{1}{X+1} + \frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2}$$

$$\int \frac{x^2}{(x+1)(x-1)^2}dx = \frac{1}{4}\int \frac{1}{X+1}dx + \frac{3}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{(x-1)^2}dx$$

$$\int \frac{x^2}{(x-1)(x-1)^2}dx = \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-1| - \frac{1}{2}\frac{1}{(x-1)} + c$$

Example:

Evaluate $\int \frac{x^2+1}{(x+2)^3}dx$

Solution:

$$\frac{x^2+1}{(x+2)^3} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{c}{(x+2)^3}$$

Multiply both side by the denominator by

$(x+2)$ $(x+2)^2$ $(x+2)^3$

$x^2 + 1 = A(x+2)^2 + B(x+2) + C$ (1)

Let $x + 2 = 0 \rightarrow x = -2.$

By substitute in (1)

$5 = A(0)^2 + B(0) + C \qquad \rightarrow c = 5$

$$\int \frac{x^2+1}{(x+2)^3}dx = \int \frac{Adx}{(x+2)} + \int \frac{Bdx}{(x+2)^2} + \int \frac{cdx}{(x+2)^3}$$

$$\int \frac{x^2+1}{(x+2)^3}dx = \int \frac{dx}{(x+2)} + \int \frac{dx}{(x+2)^2} + \int \frac{5dx}{(x+2)^3}$$

<u>Example:</u>

Evaluate

$$\int \frac{\boldsymbol{x+3}}{\boldsymbol{2x^3-8x}}\boldsymbol{dx}$$

Solution.

$$\frac{x+3}{2x^3-8x} = \frac{x+3}{2x\,(x^2-4)}$$

$$= \frac{x+3}{2x\,(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{c}{x+2}$$

Multiply both side by the denominator by

$2x\,(x-2)(x+2)$

$x + 3 = A(x-2)(x+2) + B2x(x+2) + C2x(x-2) \quad *$

Let $x = 0$ →By substitute in (*) → $A = -\frac{3}{4}$

Let $x-2=0 \rightarrow x=2 \rightarrow$ By substitute in $(*) \rightarrow B=\frac{15}{16}$

Let $x+2=0 \rightarrow x=-2 \rightarrow$ By substitute in $(*) \rightarrow C=\frac{1}{16}$

$$\int \frac{x+3}{2x^3-8x}dx = \int \frac{A}{2x}dx + \int \frac{B}{x-2}dx + \int \frac{c}{x+2}dx$$

$$\int \frac{x+3}{2x^3-8x}dx = -\frac{3}{4}\int \frac{1}{2x}dx + \frac{15}{16}\int \frac{1}{x-2}dx + \frac{1}{16}\int \frac{1}{x+2}dx$$

$$\int \frac{x+3}{2x^3-8x}dx = -\frac{3}{8}\ln|2x| + \frac{15}{16}\ln|x-2| + \frac{1}{16}\ln|x+2| + c$$

Type of improper integrals and method of evaluation

In this section we will extend the concept of a define integral to include infinite interval of integration and integrands that become infinite within the interval of integration.

Type of improper integrals:

$1-\int_{1}^{\infty} \frac{1}{x+1}dx \quad \rightarrow$ improper integral

$2-\int_{-\infty}^{4} \frac{1}{x+1}dx \quad \rightarrow$ improper integral

$3-\int_{1}^{5} \frac{1}{x-1}dx \quad \rightarrow$ improper integral

$$4-\int_{1}^{\infty}\frac{1}{x-1}dx \quad \rightarrow \text{improper integral}$$

$$1-\int_{0}^{5}\frac{1}{x}dx \quad \rightarrow \text{improper integral}$$

Information

$ln+\infty=\infty$

$ln-\infty=donote\ exisit$

$ln0^{+}=-\infty$

$ln0^{-}=donote\ exisit$

$e^{\infty}=\infty$

$e^{-\infty}=0$

$tan^{-1}+\infty=\frac{\pi}{2}$

$tan^{-1}-\infty=-\frac{\pi}{2}$

<u>Case (1): [a,+∞)</u>

$$\int_{a}^{+\infty}f(x)dx=\lim_{l\rightarrow\infty}\int_{a}^{l}f(x)dx$$

$$\lim_{l\rightarrow\infty} b=\begin{cases} number & \rightarrow converges \\ \pm\infty, not\ exisist & \rightarrow diverge \end{cases}$$

<u>Example:</u>

$$find \int_{1}^{\infty} \frac{1}{x} dx$$

Solution:

$$\int_{1}^{\infty} \frac{1}{x} dx \rightarrow \lim_{l\to\infty} \int_{1}^{l} \frac{1}{x} dx \rightarrow \lim_{l\to\infty} \left[ln|x| \;\; {}_{1}^{l} \right]$$

$$= \lim_{l\to\infty} \quad [ln|l| - ln|1|] = \lim_{l\to\infty} [\, ln|l| - 0]$$

$$= \lim_{l\to\infty} ln|l| = ln|\infty| = \infty \;\; diverge$$

Example:

$$\int_{0}^{\infty} e^{-2x}\, dx$$

Solution:

$$\int_{0}^{\infty} e^{-2x}\, dx \rightarrow \lim_{l\to\infty} \int_{0}^{l} e^{-2x} dx \rightarrow \lim_{l\to\infty} \left(\frac{e^{-2x}}{-2} \Bigg|_{0}^{l} \right)$$

$$= \lim_{l\to\infty} \left[\frac{e^{-2l}}{-2} + \frac{1}{2} \right] = \frac{e^{-2\infty}}{-2} + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2} \rightarrow converges$$

Remark:

$$e^{-2\infty} = e^{-\infty} = 0.$$

Case (2): (-∞, +∞)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

$$= \lim_{l \to -\infty} \int_{l}^{0} f(x)dx + \lim_{l \to +\infty} \int_{0}^{l} f(x)dx$$

Hence notes:

$$converges \pm converges = converges$$

$$converges \pm \ diverge \ = diverge$$

Example:

$$find \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx$$

Solution:

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{+\infty} \frac{1}{1+x^2} dx$$

$$1 - \int_{-\infty}^{0} \frac{1}{1+x^2} dx$$

$$= \lim_{l \to -\infty} \int_{l}^{0} \frac{1}{1+x^2} dx$$

$$= \lim_{l \to -\infty} \left(tan^{-1}x|_{l}^{0} \right)$$

$$= \lim_{l \to -\infty} [-tan^{-1}l]$$

$$= -tan^{-1}(-\infty) = \frac{\pi}{2}$$

$$2 - \int_{0}^{+\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{l\to+\infty} \int_0^l \frac{1}{1+x^2} dx$$

$$= \lim_{l\to+\infty} \left(tan^{-1}x|_0^l \right)$$

$$= \lim_{l\to+\infty} [tan^{-1}l - tan^{-1}0]$$

$$= \lim_{l\to+\infty} [tan^{-1}l] = tan^{-1}(+\infty) = \frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{+\infty} \frac{1}{1+x^2} dx$$

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Case (3):[a,b^-]

If f is continuous on the interval [a,b], except for an infinite discontinuity at b, then the improper integral of f over the interval [a,b],is defined as.

$$\int_a^b f(x)dx = \lim_{l\to b^-} \int_a^l f(x)dx$$

Example:

$$\int_0^1 \frac{1}{\sqrt{1-x}} dx$$

Solution:

$$\int_0^1 \frac{1}{\sqrt{1-x}} dx = \lim_{l\to\bar{1}} \int_0^l \frac{1}{\sqrt{1-x}} dx = \lim_{l\to\bar{1}} \left[-2\sqrt{1-x}\right]_0^l$$

$= \lim_{l\to 1}[-2\sqrt{1-l}+2]=2$

Case (4):[a^+,b]

If f is continuous on the interval [a,b], except for an infinite discontinuity at a, then the improper integral of f over the interval [a,b],is defined as.

$$\int_a^b f(x)dx = \lim_{l\to a+}\int_l^b f(x)dx$$

Example:

$$find \quad \int_1^2 \frac{1}{1-x}dx$$

Solution:

$$\int_0^3 \frac{1}{1-x}dx = \lim_{l\to 1^+}\int_l^2 \frac{1}{1-x}dx$$

$$= \lim_{l\to 1^+}[ln|1-x|]_l^2$$

$$= \lim_{l\to 1^+}[-ln|-1| + ln|1-l|]$$

$$= \lim_{l\to 1^+} ln|1-l| = ln|0| = -\infty$$

Example:

$$\int_0^1 lnxdx$$

Solution:

$$\int_0^1 lnx dx \rightarrow \lim_{a\to 0^+} \int_b^1 lnx dx$$

$$\lim_{a\to 0^+} (xlnx - x) \quad |_a^1$$

$$= \lim_{a\to 0^+} [(ln1 - 1) - (alna - a)]$$

$$= \lim_{a\to 0^+} [(0 - 1) - (alna - a)]$$

$$= [(-1) - (\infty)]$$

$$= (converge - diverge) = diverge$$

Example:

$$\int_0^2 \frac{1}{(x-1)^2} dx$$

Solution:

$$\int_0^2 \frac{1}{(x-1)^2} dx \rightarrow \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx$$

$$\int \quad \frac{1}{(x-1)^2} dx = \int (x-1)^{-2} dx = \frac{-1}{x-1}$$

$$\lim_{a\to 1^-} \int_0^a \frac{1}{(x-1)^2} dx + \lim_{a\to 1^+} \int_a^0 \frac{1}{(x-1)^2} dx$$

$$\lim_{a\to 1^-} \frac{-1}{x-1} \quad |_0^a + \lim_{a\to 1^+} \frac{-1}{x-1} \quad |_a^2$$

$$\lim_{a\to 1^-} \left(\frac{-a}{a-1} - \frac{-0}{0-1}\right) + \lim_{a\to 1^+} \left(\frac{-2}{2-1} - \frac{-a}{a-1}\right)$$

$$\lim_{a\to1^-}\left(\frac{-a}{a-1}-0\right)+\lim_{a\to1^+}\left(-2-\frac{-a}{a-1}\right)$$

$$diverge + diverge = diverge$$

Exercise:

$$1-find\ \int_1^{\infty}\frac{1}{x^3}dx$$

$$2-find\int_0^{3}\frac{1}{x-3}dx$$

$$3-find\int_1^{4}\frac{1}{(x-2)^{2/3}}dx$$

$$4-find\int_0^{+\infty}\frac{1}{\sqrt{2}(x+1)}dx$$

$$5-find\int_0^{2}\frac{1}{(x-1)^2}dx$$

Sequences and their limit, monotone sequences

Sequences and their limit:

Definition:

A sequences is a function whose domain is a set of positive integers. Specifically,

we will regard the expression $\{a_n\}_{n=1}^{+\infty}$ be an alternative notion for the function $f(n) = a_n, \quad n = 1,2,3, \dots \dots$

There are two types of sequence

1-The first type is finite sequence

2-The second type is infinite sequence.

Example:

$a_n = \{1,2,3\}$ finite sequence

$a_n = \{1,2,3, \dots .\}$ infinite sequence

Example:

Find the three terms of the sequence.

$$a_n = \frac{5n}{4+n^2}$$

Solution:

$$n_1 = 1 \rightarrow a_1 = \frac{5n}{4+n^2} = \frac{5(1)}{4+(1)^2} = \frac{5}{5} = 1$$

$$n_2 = 2 \rightarrow a_2 = \frac{5n}{4+n^2} = \frac{5(2)}{4+(2)^2} = \frac{10}{8} = \frac{5}{4}$$

$$n_3 = 3 \rightarrow a_3 = \frac{5n}{4+n^2} = \frac{5(3)}{4+(3)^2} = \frac{15}{13}.$$

The three terms of the sequence are $1, \frac{5}{4}, \frac{15}{13}$

Example:

Find the four terms of the sequence.

$$a_n = cos\left(\frac{n\pi}{2}\right)$$

Solution:

$$n_1 = 1 \rightarrow a_1 = cos\left(\frac{n\pi}{2}\right) = cos\left(\frac{\pi}{2}\right) = 0.$$

$$n_2 = 2 \rightarrow a_2 = cos\left(\frac{n\pi}{2}\right) = cos\left(\frac{2\pi}{2}\right) = -1.$$

$$n_3 = 3 \rightarrow a_3 = cos\left(\frac{n\pi}{2}\right) = cos\left(\frac{3\pi}{2}\right) = 0.$$

$$n_4 = 4 \rightarrow a_4 = cos\left(\frac{n\pi}{2}\right) = cos\left(\frac{4\pi}{2}\right) = 1.$$

The four terms of the sequence are 0,-1,0,1

Example:

Find the four terms of the sequence.

$$a_n = (-1)^{n+1} tan\left(\frac{n\pi}{2}\right)$$

Solution:

$$n_1 = 1 \rightarrow a_1 = (-1)^{n+1} tan\left(\frac{n\pi}{2}\right) = tan\left(\frac{\pi}{2}\right) = \frac{\sin 90}{\cos 90}$$

$$= \frac{1}{0} = \infty.$$

$$n_2 = 2 \rightarrow a_2 = (-1)^{n+1} tan\left(\frac{n\pi}{2}\right) = -tan\left(\frac{2\pi}{2}\right) = 0.$$

$$n_3 = 3 \rightarrow a_3 = (-1)^{n+1} tan\left(\frac{n\pi}{2}\right) = tan\left(\frac{3\pi}{2}\right) = \infty.$$

$$n_4 = 4 \rightarrow a_4 = (-1)^{n+1} tan\left(\frac{n\pi}{2}\right) = -tan\left(\frac{4\pi}{2}\right) = 0.$$

The four terms of the sequence are ∞,0, ∞,0

Example:

Find the limit of the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

Solution:

$$\lim_{n\to+\infty} \frac{n}{2n+1} = \lim_{n\to+\infty} \frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2 + \frac{1}{\infty}}$$

$$= \frac{1}{2+0} = \frac{1}{2}$$

Exercise:

1-Find the limit of the sequence $\lim_{n\to+\infty} \sqrt[n]{n}$

2-Find the four terms of the sequence.

$$a_n = sin\left(\frac{n\pi}{2}\right)$$

3- Find the limit of the sequence $\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}$

4- Find the limit of the sequence $\left\{\frac{3n}{2n}\right\}_{n=1}^{+\infty}$

Remark:

$1-)\ -1 \leq sin\theta \leq 1$

$2-)\ -1 \leq cos\theta \leq 1$

$3-)\ \infty.0 = 0$

$4-)\ \frac{n}{\infty} = 0$

Example:

Determine whether the sequence converge or diverges.

$$\lim_{n\to\infty} \frac{1}{n} cos\left(\frac{n}{2}\right)$$

Solution:

$$-1 \leq cos\left(\frac{n}{2}\right) \leq 1$$

$$\frac{-1}{n} \le cos\left(\frac{n}{2}\right) \le \frac{1}{n}$$

$$1- \lim_{n\to\infty} \quad \frac{-1}{n} = \frac{-1}{\infty} = 0$$

$$2- \lim_{n\to\infty} \quad \frac{1}{n} = \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty} \quad \frac{1}{n} cos\left(\frac{n}{2}\right) = 0. \quad convergent$$

Example:

Determine whether the sequence converge or diverges.

$$\lim_{n\to\infty} \quad \frac{sin2n}{1+\sqrt{n}}$$

Solution:

$$-1 \le sin2n \le 1$$

$$\frac{-1}{1+\sqrt{n}} \le \frac{sin2n}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}$$

$$1- \lim_{n\to\infty} \quad \frac{-1}{1+\sqrt{n}} = \frac{-1}{\infty} = 0$$

$$2- \lim_{n\to\infty} \quad \frac{1}{1+\sqrt{n}} = \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty} \quad \frac{sin2n}{1+\sqrt{n}} = 0. \quad convergent$$

monotone sequences:

In this section we will study several techniques that can be used to determine whether a sequences converges.

Definition:

A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $\quad a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

increasing if $\quad a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$

srrictily decreasing if $\quad a_1 > a_2 > a_3 > \cdots > a_n > \cdots$

decreasing if $\quad a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

Example:

Determine the monotone sequence

1,1,2,2,3,3…… increasing

$1,1,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3}\ldots..$ decreasing

$\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{1}{3}\ldots..\frac{n}{n+1}$ strictly increasing

$1,\frac{1}{2},\frac{1}{3}\ldots..\frac{1}{n}$ srrictily decreasing

$1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4}\ldots(-1)^{n+1}\frac{1}{n}$ $Neither$ increasing nor decreasing

9. Infinite series . The comparison

Definition:

An infinite series is an expression that can be written in the form $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$

The numbers $u_1, u_2, u_3, \ldots$ are called the terms of the series.

Definition:

Let $\{s_n\}$be a sequence of the partial sums of the series

$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$

If the sequence $\{s_n\}$converges to a limit s, then the series is said to converges to s, and s is called the sum of the series. we denote this by writing

$$s = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to be diverge. A diverge series has no sum.

Example:

Determine whether the series $\sum_{n=1}^{\infty} (-1)^n$ converge or diverge. If it converges, find the sum.

Solution:

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_3 = 1 - 1 + 1 - 1 = 0$

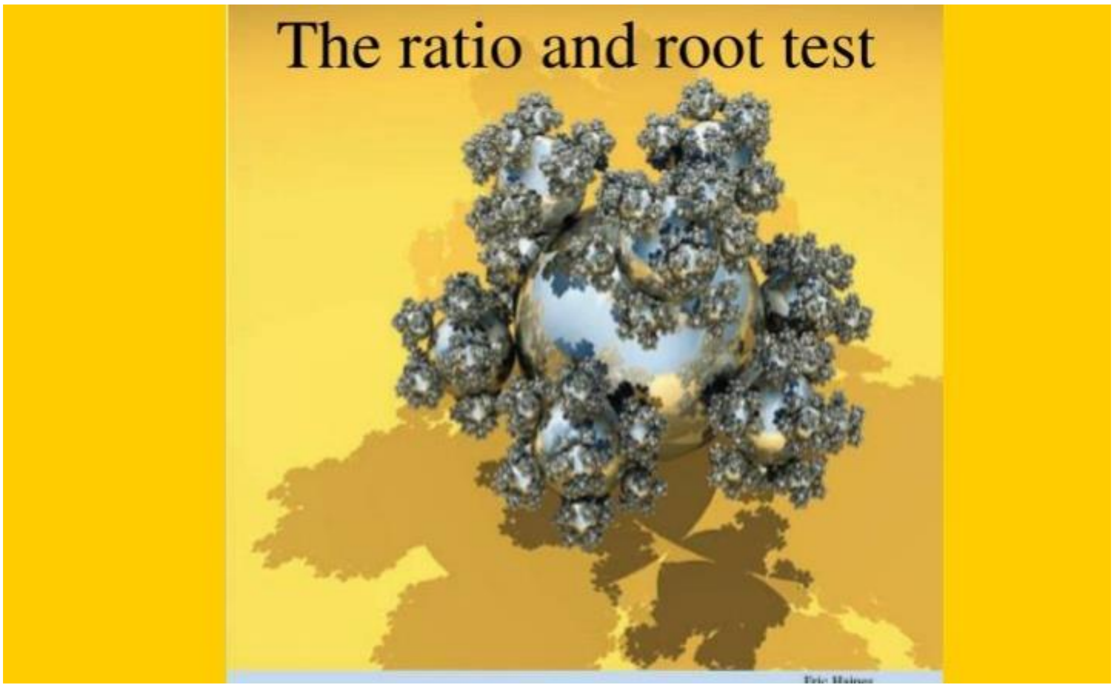
And so.

Thus. The sequence of partial sum is

1,0,1,0,……

Since This is diverge sequence, the given series diverges and consequently has no sum.

10. Ratio and Root tests. Alternating series



Definition:

Let $\sum \; u_k$ be a series with positive terms and suppose that $p = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$

(a) *series converges if* $p < 1$.

(b) *series diverges if* $p > 1$ *or* $p = +\infty.$

(c) *the test is inconcusive if* $p = 1.$

Example:

Test the following series by ratio test. $\sum_{k=1}^{\infty} \frac{2^k}{k^2}$

Solution:

$$p = \lim_{k\to\infty} \left| \frac{\frac{2^{k+1}}{(k+1)^2}}{\frac{2^k}{k^2}} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+1)^2} \cdot \frac{k^2}{2^k} \right|$$

$$\lim_{k\to\infty} \frac{2k^2}{(k+1)^2} = \lim_{k\to\infty} \frac{2k^2}{k^2+2k+1} = 2 > 1. \ \textit{series diverges}$$

By: (b) *series diverges if* $p > 1$ *or* $p = +\infty.$

Example:

Test the following series by ratio test. $\sum_{k=1}^{\infty} \frac{1}{(2k+1)!}$

Solution:

Example:

Test the following series by ratio test. $\sum_{k=1}^{\infty} \frac{1}{k}$

Solution:

Definition:

Let $\sum \quad u_k$ be a series with positive terms and suppose that $p = \lim_{k\to+\infty} \sqrt[k]{u_k}$

(a) $series\ converges\ if\ p < 1.$

(b) $series\ diverges\ if\ p > 1or\ p = +\infty.$

(c) $the\ test\ is\ inconcusive\ if\ \ p = 1.$

Example:

Test the following series by ratio test. $\sum_{k=1}^{\infty} k^k$

Solution:

$p = \lim_{k\to+\infty} \sqrt[k]{k_k} = \lim_{k\to+\infty} k \cong \infty.\ diverges$

By: (b) $series\ diverges\ if\ p > 1or\ p = +\infty.$

Definition:

If $a_k > 0\ for\ k = 1,2,3, \dots., then\ the\ series$

$a_1 - a_2 + a_3 - a_4\ + \cdots \quad 0r$

$-a_1 + a_2 - a_3 + a_4$
$- \cdots\ the\ series\ is\ called\ alternating\ series$

Convergent if the following conditional hold:

$(a)|\ \ a_1| \geq |a_2| \geq a_3 \geq |a_4\ | + \cdots$

$(b)\ \ a_{2m+1} \leq 0\ \ , \qquad a_{2m}\ \ \geq 0.\ \ \forall m = 1,2,3, ..$

$(c) \lim_{k\to+\infty} a_k = 0.$

Example:

Test the following series is converge or diverge

$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$

Solution:

$$(a)-\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$$

$$|a_1|=|-1|=1$$

$$|a_2|=\left|+\frac{1}{2}\right|=\frac{1}{2}$$

$$|a_3|=\left|-\frac{1}{3}\right|=\frac{1}{3}$$

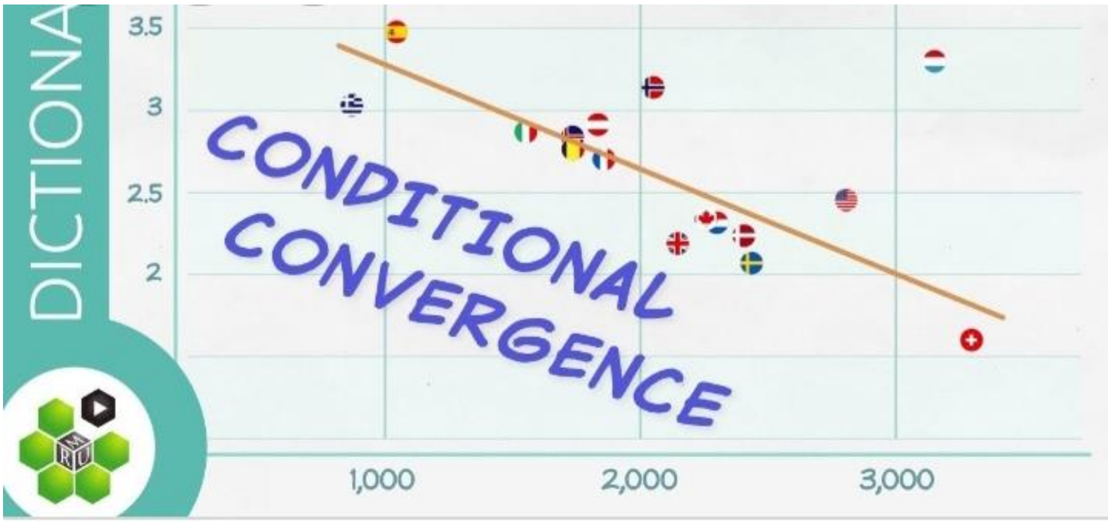
$$\therefore \quad 1>\frac{1}{2}>\frac{1}{3}.$$

$$(b)-(b) \quad a_1=-1<0 \ , \qquad a_2=\frac{1}{2} \ >0.$$

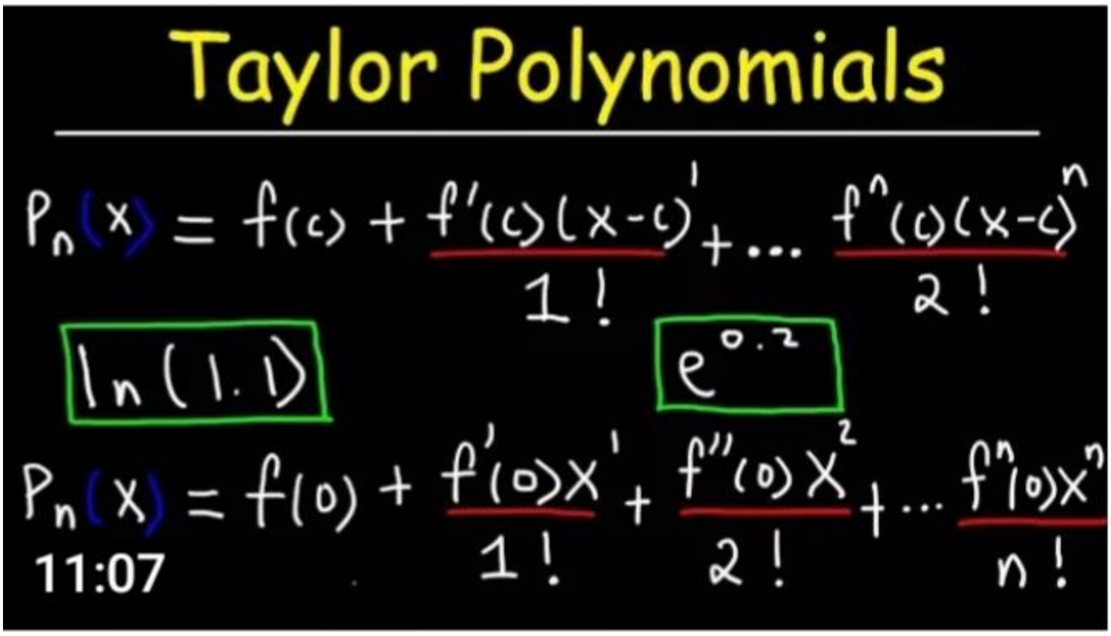
$$(c) \lim_{k\to+\infty}(-1)^k \frac{1}{k}=0.$$

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k} \ is\ converges.$$

11. Conditional converges.



Maclaurin series and taylor series. And their approximation power series.



Definition:

The Taylor series for f at $x = a$ is a series of the form.

$$T(a) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^k(a)}{k!}(x-a)^k$$

Example:

Find the Taylor series for the function $f(x) = e^x \ when \ a = 10$

Solution:

$f(x) = e^x \rightarrow f(10) = e^{10}$

$f'(x) = e^x \rightarrow f'(10) = e^{10}$

$f''(x) = e^x \rightarrow f''(10) = e^{10}$

$$T(a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots + \frac{f^k(a)}{k!}(x-a)^k$$

$$T(a) = e^{10} + e^{10}(x-10) + \frac{e^{10}}{2!}(x-10)^2 + \frac{e^{10}}{3!}(x-10)^3 + \cdots$$

$$T(a) = e^{10}\left[(x-10) + \frac{1}{2!}(x-10)^2 + \frac{1}{3!}(x-10)^3 + \cdots\right]$$

$$T(a) = e^{10}\sum_{k=0}^{\infty}\frac{(x-10)^k}{k!}$$

Example:

Find the Taylor series for the function $f(x) = lnx \;\; when \; a = 1$

Solution:

$$f(x) = lnx \rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(1) = 2$$

$$T(a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots + \frac{f^k(a)}{k!}(x-a)^k$$

$$T(a) = 0 + (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots$$

$$T(a) = \sum_{k=1}^{\infty} \frac{(-1)^{2k} k (x-1)^k}{k!}$$

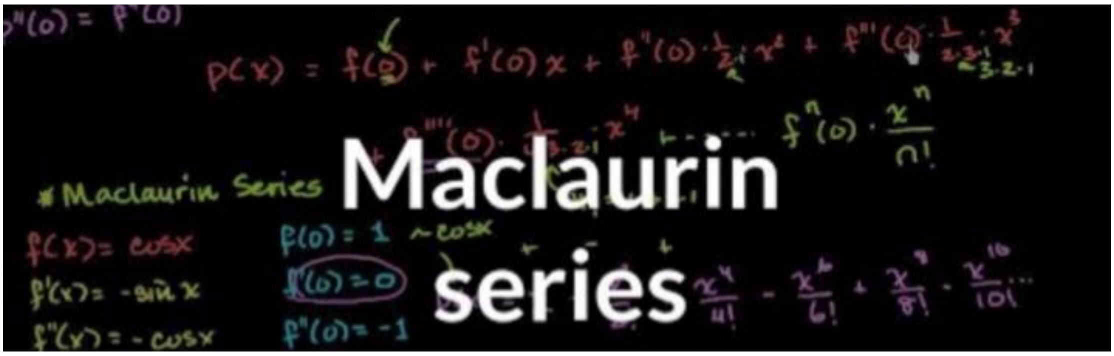
Exercise:

Find the Taylor series for the following functions

$1 - f(x) = \frac{1}{x} \ \ when\ a = -1$

$$2 - f(x) = cosx \ \ when\ a = \frac{-\pi}{4}$$

$$3 - f(x) = x^2 \ \ when\ a = \frac{1}{2}$$



Definition:

The Maclaurin series for f at $x = 0$ is a series of the form.

$$T(0) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x)^k = f(a) + f'(0)(x) + \frac{f''(a)}{2!}(x)^2 + \cdots + \frac{f^k(a)}{k!}(x)^k$$

Example:

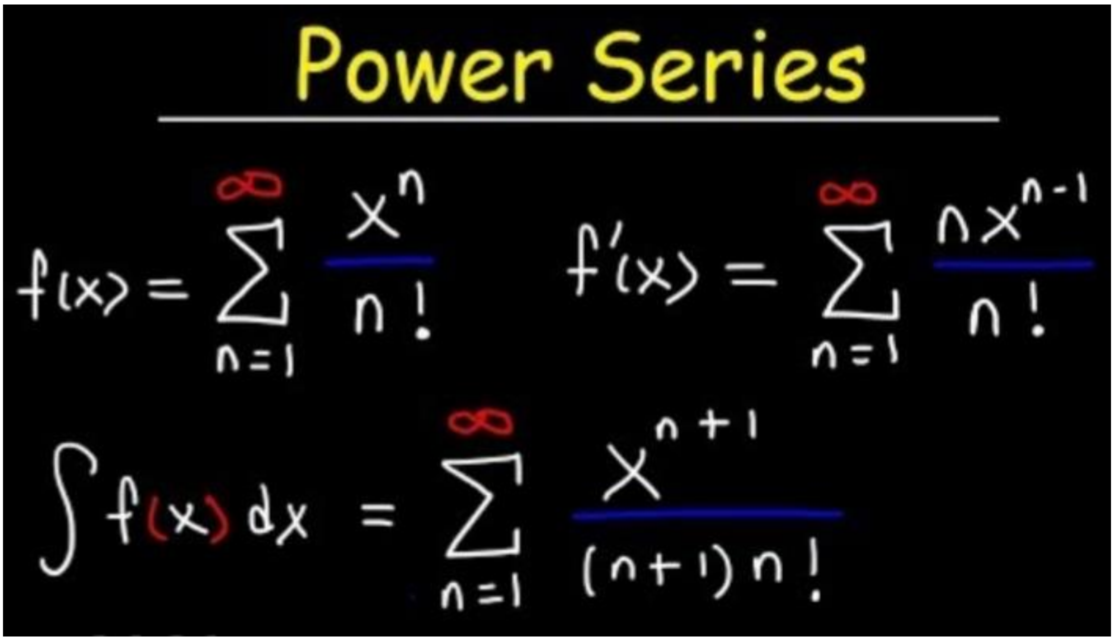
Find the Maclaurin series for the following function

$f(x) = (1+x)^{\frac{3}{2}}$

Solution:

$f(x) = (1+x)^{\frac{3}{2}} \to f(0) = 1$

12- Differentiating and integrating power series.



Definition:

A power series is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$$

Or

$$\sum_{k=0}^{\infty} c_k(x-a_k)^k = a_0 + c_1(x-a_1) \quad + c_2(x-a_2)^2$$
$$+ \cdots + c_k(x-a_k)^k + \cdots$$

Example:

$find\ the\ second\ terms\ of\ \ a\ power\ series\ in\ x, \sum_{k=1}^{\infty} \frac{x^k}{k}$

Solution:

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots is\ a\ power\ series\ in\ x$$

Example:

$find\ the\ thired\ terms\ of\ \ a\ power\ series\ in\ x, \sum_{k=1}^{\infty} \frac{x^k}{k!}$

Solution:

$$\sum_{k=1}^{\infty} \frac{x^k}{k!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots is\ apower\ series\ in\ x$$

Example:

$find\ the\ thired\ terms\ of\ \ a\ power\ series\ in\ (x-2),$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(x-2)^k}{k!}$$

Solution:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{k!} = 1 - \frac{x-2}{1} + 0 + \cdots \text{is apower series in } (x-2).$$

Remark:

$$1 - sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

$$1 - cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$

Differentiating power series.

Definition:

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of converges c then:

$$a - \sum_{k=0}^{\infty} k a_k x^{k-1} \text{ also has radies of convergence } c$$

$$b - f(x) \text{is differentiable on } (-c, c).$$

$$c - f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} \text{ on } (-c, c).$$

Integration power series.

Definition:

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of converges c then:

$$a-\sum_{k=0}^{\infty}\frac{a_k x^{k+1}}{k+1}\ \ also\ has\ radies\ of\ convergence\ \ c$$

$$b-\int\quad f(x)dx\ \ is\ differentiable\ on\ (-c,c).$$

$$c-\int\quad f(x)dx=\sum_{k=0}^{\infty}\frac{a_k x^{k+1}}{k+1}+c\qquad on\quad (-c,c).$$

Example:

Prove that $\frac{d}{dx}\ sinx = cosx$

Solution:

$$sinx = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\dots$$

$$\frac{d}{dx}sinx = 1-3\frac{x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}\dots$$

$$\frac{d}{dx}sinx = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\ \ = cosx.$$

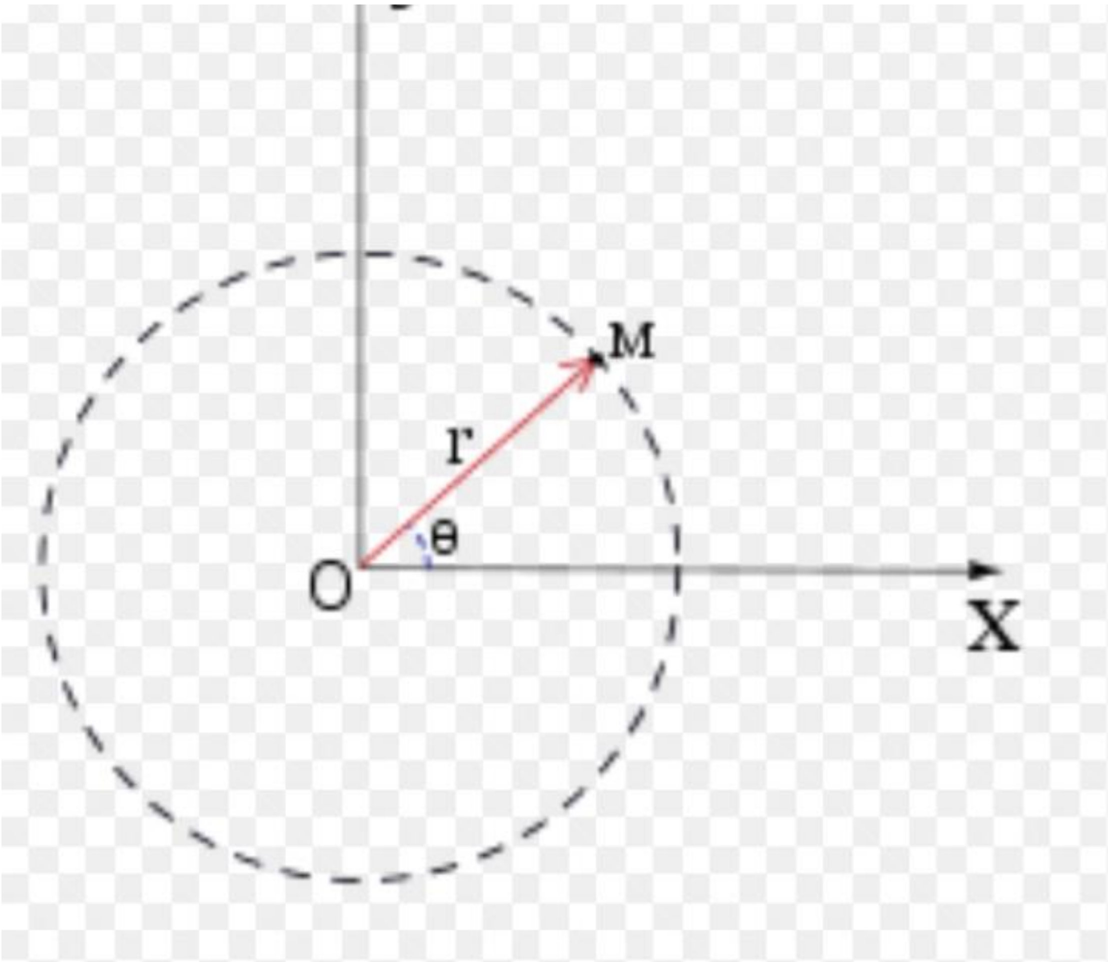
Exc

Prove that $\int cosx = -sinx + c, \quad -\infty < x < \infty.$

Example:

Express $\int sin\ x^2\, dx\ \ as\ a\ power\ series$

13-polar coordinates. Curves defined by parametric equations.



Def

A polar coordinates system in a plane consists of a fixed point o, called the pole (or origin),and a ray emantig from the pole. called the polar axis.

polar coordinates (r, θ^0) which the first number r gives the directed distance from 0 to p. and the second number θ gives the directed angle from the initial ray to op.

Example:

Find the polar coordinates of the point $(3, 30^0)$.

Slo

$p(3,30)$

$p(3,30-360)$

$p(-3,30-180)$

$p(-3,30+180)$

Exce:

Find the polar coordinates of the points

$1-p(5, 45^0)$

$2-p(4, 90^0)$,

$3-p(6, 270^0)$,

Example:

Find all the polar coordinates of the point $(2, 30^0)$.

Solution:

1-when $r=2$

the angle $\theta = 30 \pm 1.360^0$

the angle $\theta = 30 \pm 2.360^0$

the angle $\theta = 30 \pm 3.360^0$

$\vdots$

There fore the polar coordinates

$(2,30 \pm \mathrm{n}.360^0)\ n = 0,1,2,3,\ldots$

$or\ \ (2,30 + \mathrm{n}.360^0)\ n = 0,\pm1,\pm2,\pm3,\ldots$

2-when $r = -2$

the angle $\theta = -150 \pm 1.360^0$

the angle $\theta = -150 \pm 2.360^0$

the angle $\theta = -150 \pm 3.360^0$

$\vdots$

There fore the polar coordinates

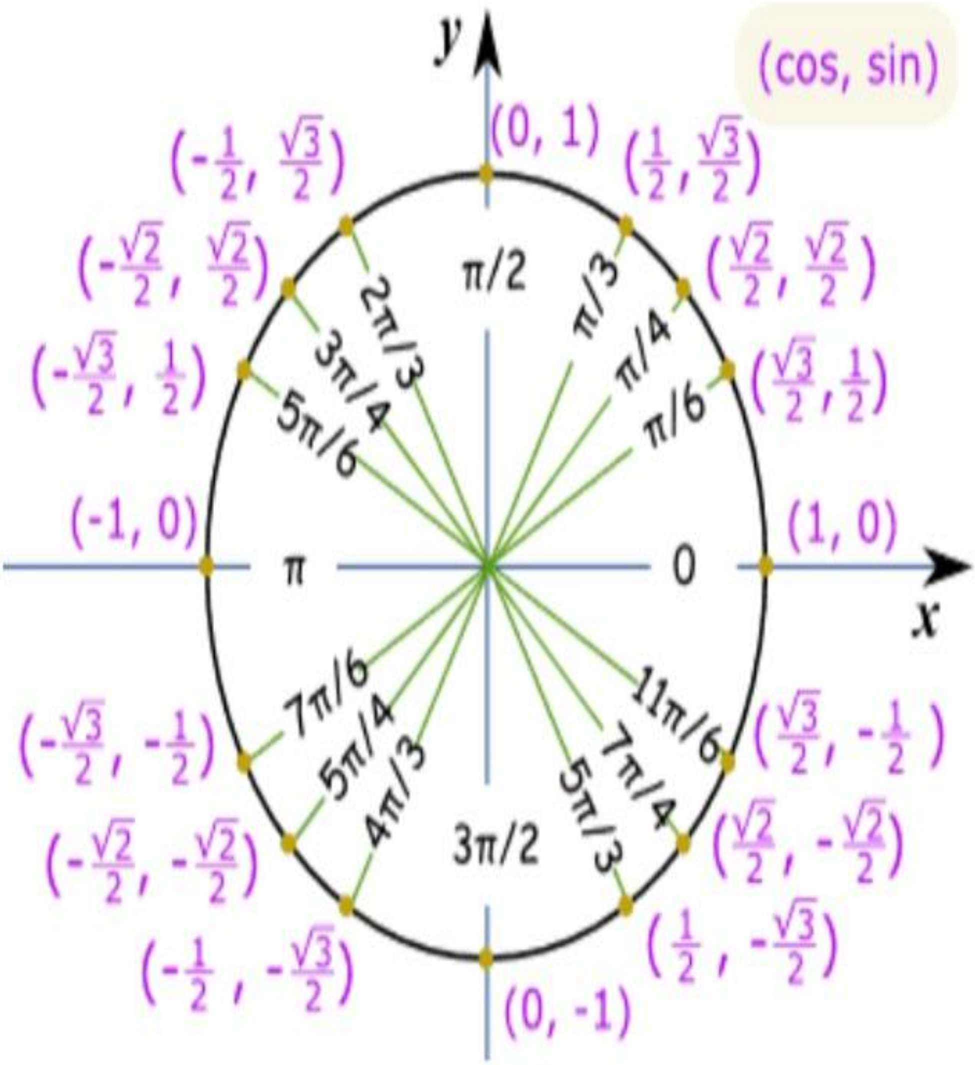
$(-2,-150 \pm \mathrm{n}.360^0)\ n = 0,1,2,3,\ldots$

$or\ \ (-2,-150 + \mathrm{n}.360^0)\ n = 0,\pm1,\pm2,\pm3,\ldots$

Thus the polar coordinates are

$(2,30 + \mathrm{n}.360^0)\ n = 0,\pm1,\pm2,\pm3,\ldots$

$(-2,-150 + \mathrm{n}.360^0)\ n = 0,\pm1,\pm2,\pm3,\ldots$



Remark:

To find the Cartesian coordinate equivalent to coordinate and rice verse, we use the following equation

$1 - r^2 = r^2cos^2\theta + r^2sin^2\theta$

$2 - cos\theta = \frac{x}{r}$

$3 - sin\theta = \frac{y}{r}$

$4 - tan\theta = \frac{y}{x}$

$5 - x = rcos\theta$

$6 - y = rsin\theta$

$7 - \quad r = \sqrt{x^2 + y^2}$

$8 - \theta = cos^{-1}\frac{x}{r}$

$9 - \theta = sin^{-1}\frac{y}{r}$

Ex

Find the Cartesian coordinate of the point $p(2, 60^0)$.

Sol

$x = rcos\theta \rightarrow x = 2\cos \quad 60 \rightarrow x = 2\frac{1}{2} \rightarrow x = 1$

$y = rsin\theta \rightarrow y = 2\sin \quad 60 \rightarrow y = 2\frac{\sqrt{3}}{2} \rightarrow y = \sqrt{3}$

$(x, y) = \left(1, \sqrt{3}\right)$

Ex

Find the Cartesian coordinate of the point $p(2, 90^0)$.

Sol

$x = rcos\theta \rightarrow x = 2\cos \quad 90 \rightarrow x = 2.0 \rightarrow x = 0$

$y = rsin\theta \rightarrow y = 2\sin \quad 90 \rightarrow y = 2.1 \rightarrow y = 2$

$(x, y) = (0,2)$

Exc

Find the Cartesian coordinate of the points and sketch.

$1 - p(-2, 0^0)$.

$2 - p(-2, 90^0)$.

$3 - p\left(-2, \frac{\pi}{6}\right)$.

$4 - p\left(-4, \frac{\pi}{3}\right)$.

14-Tangent line and length for parametric and polar curves. Area in polar coordinates.

In this section we will derive the formula required to find slopes, tangent lines, and arc lengths of parametric and polar curves.

Def736

We will be concerned in this section with curves that are given by parametric equations

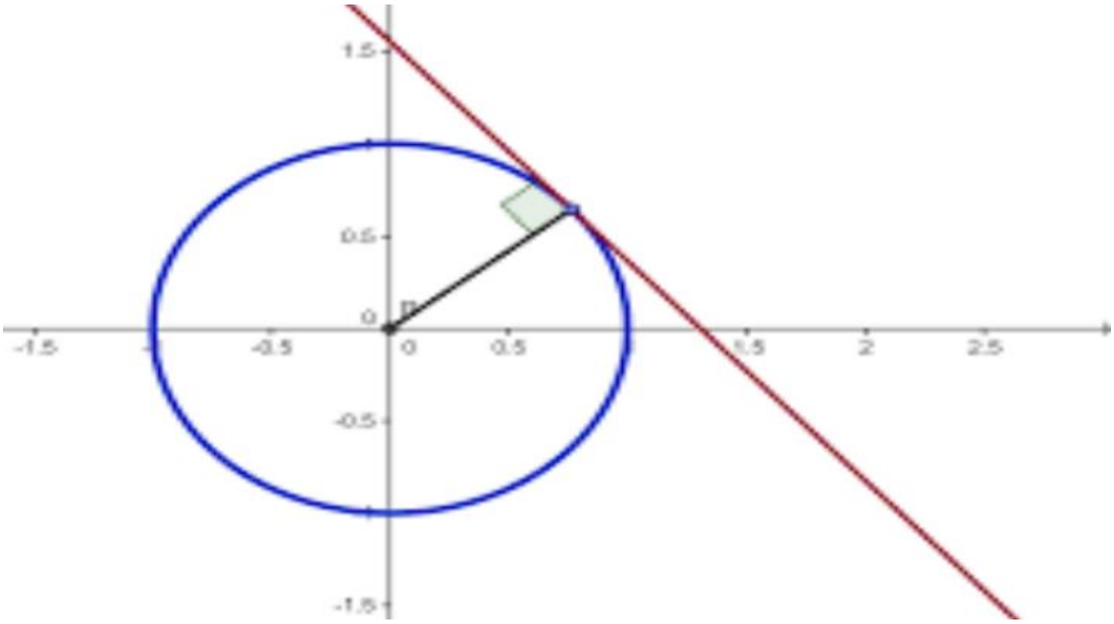
$$x = f(t) \quad , \quad y = g(t).$$

In which $f(t)$ and $g(t)$ have continuous first derivative with respect to t.it can be proved that if $dx/dt \neq 0$, then y is differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

This formula makes it possible to find $\frac{dy}{dx}$ directly from the parametric equations without eliminating the parameter.

Ex



Find the slope and tangent line to the unit circle.

$x = \cos t, \quad y = \sin t\,, \qquad 0 \le t \le 2\pi$

At the point where $t = {}^{\pi}/_{6}$

<u>Solution:</u>

$$By: \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad\qquad (*)$$

The slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{cost}{-\sin t} = -cost$$

Thus, the slope at $t = {}^{\pi}/_{6}$ is

$$\frac{dy}{dx}\ |_{t=\pi/6} = -\cos \pi/6 = -\sqrt{3}$$

Ex

With out eliminating the parameter.

Find $\frac{dy}{dx}$ and d^2y/d^2x at the points (1,1) and (1,-1) on the semicubical parabola given by the parametric equations

$x = t^2\ , \quad y = t^3\ , (-\infty < t < +\infty)$

Sol

$$By: \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t. \quad t \neq 0 \qquad (1)$$

And from (*) applied to $y' = \frac{dy}{dx}$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dy}{dt}} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}. \qquad (2)$$

Since the point (1,1) on the curve corresponds to t=1 in the parametric equations, it follows from (1) and (2) that

$$\frac{dy}{dx}\ |_{t=1} = \frac{3}{2} \qquad and$$

$$\frac{d^2y}{dx^2}\ |_{t=1} = \frac{3}{4}$$

Similarly, the point (1,-1) on the curve corresponds to t=-1 in the parametric equations, so applying (1) and (2) again yields

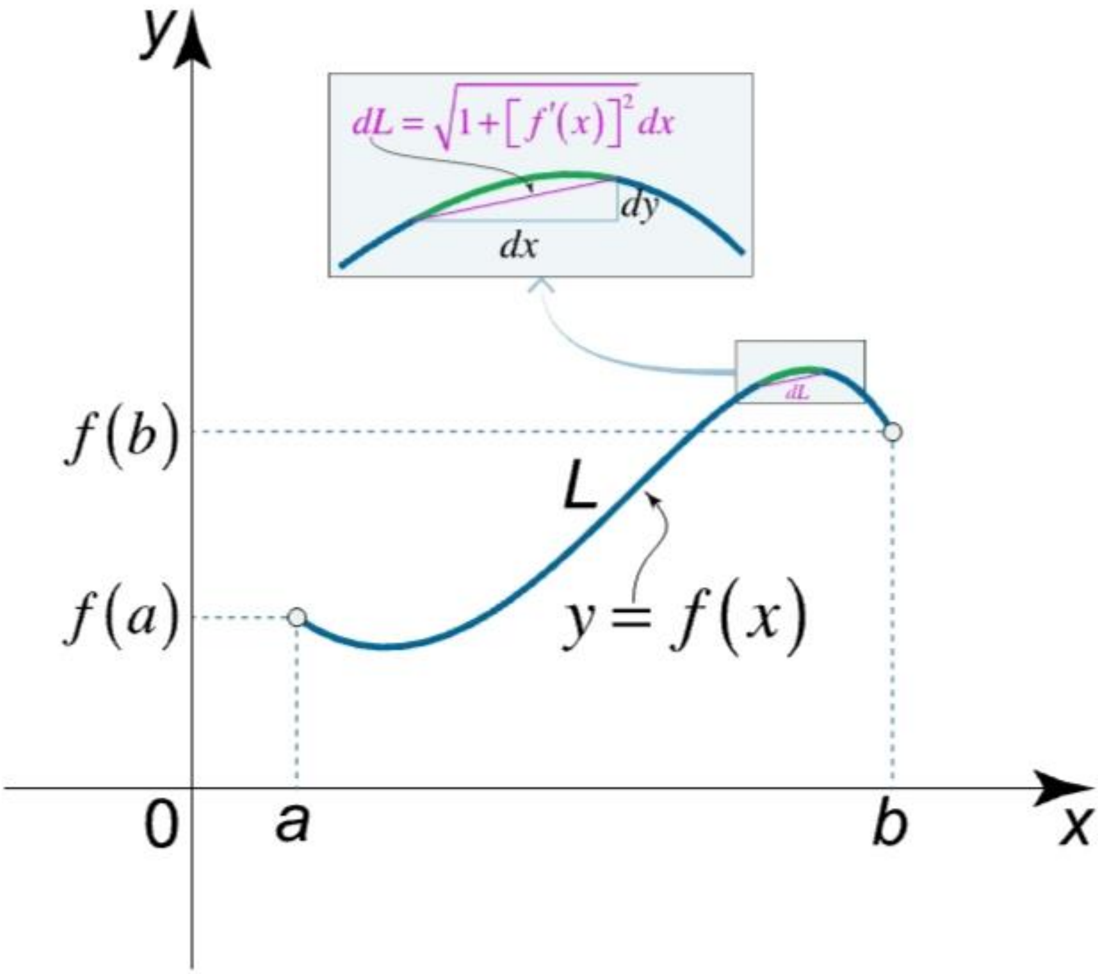
$$\frac{dy}{dx} \Big|_{t=-1} = -\frac{3}{2} \qquad and$$

$$\frac{d^2y}{dx^2} \Big|_{t=-1} = \frac{-3}{4}$$

Def 740ca

If no segment of the polar curve $r = f(\theta)$is traced more than once as θ increases from α to $\beta$and if ${dr}/{d\theta}$is continuous for $\alpha \leq \theta \leq \beta$, then the arc length L from $\theta = \alpha\ to\ \theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



Ex

Find the arc length of the spiral $r = e^{\theta}$ in Figure between $\theta = 0$ and $\theta = \pi$.

Sol

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2 d\theta}$$

$$L = \int_0^{\pi} \sqrt{2}\, e^{\theta} d\theta = \sqrt{2}\, e^{\theta} \quad]_0^{\pi} = \sqrt{2}(\, e^{\theta} - 1) \approx 31.3$$

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